

Using the tutorial Class

The `tutorial` class provides a simple and elegant way to create professional tutorial sheets, assignments, and problem sets.

1. Loading the Class

Load the class in the usual way:

```
1 \documentclass{tutorial}
```

To print solutions, use the `[solutions]` option:

```
1 \documentclass[solutions]{tutorial}
```

2. Setting Course Information

Before `\begin{document}`, specify the course information.

```
1 \CourseCode{PH31207}
2 \CourseTitle{Classical Mechanics}
3 \Semester{Autumn 2026}
4
5 \TutorialNo{01}
6
7 \Instructor{John Doe}
8 \Institute{Department of Physics\\
9 Indian Institute of Technology Kharagpur}
10
11 \Date{26 July 2026}
12 \Time{11:00 a.m.}
```

3. Creating the Title

After beginning the document, generate the title using

```
1 \begin{document}
2
3 \thispagestyle{firstpage}
4
5 \maketutorialtitle
```

4. Creating Problems

Problems are created using the problem environment.

```
1 \begin{problem}
2
3 State Newton's laws of motion.
4
5 \end{problem}
```

Problems are numbered automatically.

5. Displaying Marks

An optional argument specifies the marks assigned to the problem.

```

1 \begin{problem}[10 Marks]
2
3 Derive the Euler--Lagrange equation.
4
5 \end{problem}

```

6. Creating Subproblems

Use the subproblems environment for multipart questions.

```

1 \begin{problem}[12 Marks]
2
3 Answer the following.
4
5 \begin{subproblems}
6
7 \item Write the Lagrangian.
8
9 \item Derive the equations of motion.
10
11 \item Discuss the conserved quantities.
12
13 \end{subproblems}
14
15 \end{problem}

```

The first level is labelled

(a) , (b) , (c) , ...

Nested subproblems are also supported.

7. Writing Solutions

Solutions are written using the solution environment.

```

1 \begin{solution}
2
3 The Euler--Lagrange equation follows
4 from Hamilton's principle.
5
6 \end{solution}

```

Solutions are hidden by default.

To print all solutions, compile the document using

```

1 \documentclass[solutions]{tutorial}

```

No changes to the source document are required.

8. Mathematics

The class is intended to be compiled with **XeLaTeX** or **LuaLaTeX** and supports Unicode mathematics via the unicode-math package.

Inline mathematics:

```
1 $a^2+b^2=c^2$
```

Displayed equations:

```
1 \begin{equation}
2 F=ma.
3 \end{equation}
```

Aligned equations:

```
1 \begin{align}
2 E &= mc^2, \\
3 F &= ma.
4 \end{align}
```

Grouped equations:

```
1 \begin{subequations}
2 \begin{align}
3 x &= r \sin \theta \cos \phi, \\
4 y &= r \sin \theta \sin \phi, \\
5 z &= r \cos \theta.
6 \end{align}
7 \end{subequations}
```

9. Cross Referencing

Problems and equations may be labelled and referenced in the standard LaTeX manner.

```
1 \begin{problem} \label{prob:newton}
2
3 ...
4
5 \end{problem}
```

Later in the document,

```
1 Problem ~\ref{prob:newton}
```

will refer to the numbered problem.

10. Complete Example

```

1 \documentclass[solutions]{tutorial}
2
3 \CourseCode{PH31207}
4 \CourseTitle{Classical Mechanics}
5 \Semester{Autumn 2026}
6 \TutorialNo{01}
7
8 \Instructor{John Doe}
9 \Date{26 July 2026}
10 \Time{11:00 a.m.}
11
12 \begin{document}
13
14 \thispagestyle{firstpage}
15
16 \maketutorialtitle
17
18 \begin{problem}[10 Marks]
19
20 State Newton's second law.
21
22 \end{problem}
23
24 \begin{solution}
25
26 Newton's second law is
27
28 \[
29 \mathbf{F}=m\mathbf{a}.
30 \]
31
32 \end{solution}
33
34 \end{document}

```

Compile the document using **XeLaTeX** or **LuaLaTeX** to obtain the best typography.

Problem 1

8 Marks

Starting from Hamilton's principle, derive the Euler–Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = 0. \quad (1.1)$$

Hence show that, for the Lagrangian

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V(x), \quad (1.2)$$

the Euler–Lagrange equation reduces to Newton's second law.

Solution

Hamilton's principle states that the action

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) dt \quad (1.3)$$

is stationary under arbitrary variations of the path satisfying

$$\delta q(t_1) = \delta q(t_2) = 0.$$

Therefore,

$$\delta S = 0. \quad (1.4)$$

The variation of the action is

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \right) dt. \quad (1.5)$$

Since

$$\delta \dot{q} = \frac{d}{dt} (\delta q),$$

we obtain

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{d}{dt} (\delta q) \right) dt. \quad (1.6)$$

Integrating the second term by parts,

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{d}{dt} (\delta q) dt = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q dt. \quad (1.7)$$

The boundary term vanishes because

$$\delta q(t_1) = \delta q(t_2) = 0.$$

Hence,

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right] \delta q dt. \quad (1.8)$$

Since the variation δq is arbitrary, the integrand must vanish, giving the Euler–Lagrange equation,

$$\boxed{\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = 0.} \quad (1.9)$$

Now consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V(x). \quad (1.10)$$

We compute

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x}, \quad (1.11)$$

so that

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = m \ddot{x}. \quad (1.12)$$

Also,

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{dV}{dx}. \quad (1.13)$$

Substituting into the Euler–Lagrange equation,

$$m \ddot{x} + \frac{dV}{dx} = 0, \quad (1.14)$$

or equivalently,

$$m \ddot{x} = -\frac{dV}{dx}. \quad (1.15)$$

Since the conservative force is given by

$$F(x) = -\frac{dV}{dx}, \quad (1.16)$$

we obtain

$$\boxed{F = m\ddot{x}}, \quad (1.17)$$

which is precisely Newton's second law.

Problem 2

12 Marks

A particle of mass m moves under a central force $\vec{F} = F(r) \hat{r}$.

- (a) Obtain the Lagrangian in plane polar coordinates.
- (b) Derive the equations of motion.
- (c) Show that the angular momentum $\vec{L}$ is conserved.
- (d) Derive the effective potential $V_{\text{eff}}(r)$ and discuss its physical significance.

Solution

- (a) The kinetic energy of the particle in plane polar coordinates is

$$T = \frac{1}{2}m \left(\dot{r}^2 + r^2\dot{\theta}^2 \right). \quad (2.1)$$

Since the force is conservative,

$$\vec{F} = -\nabla V(r), \quad (2.2)$$

where

$$F(r) = -\frac{dV}{dr}. \quad (2.3)$$

Hence the Lagrangian is

$$\boxed{\mathcal{L} = \frac{1}{2}m \left(\dot{r}^2 + r^2\dot{\theta}^2 \right) - V(r)}. \quad (2.4)$$

- (b) The generalized coordinates are r and θ .

Applying the Euler–Lagrange equation to the radial coordinate,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) - \frac{\partial \mathcal{L}}{\partial r} = 0, \quad (2.5)$$

we obtain

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{dV}{dr} = 0, \quad (2.6)$$

or

$$\boxed{m\ddot{r} = mr\dot{\theta}^2 - \frac{dV}{dr}}. \quad (2.7)$$

Similarly, for the angular coordinate,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0. \quad (2.8)$$

Since the Lagrangian is independent of θ ,

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0, \quad (2.9)$$

and therefore

$$\boxed{\frac{d}{dt} (mr^2\dot{\theta}) = 0.} \quad (2.10)$$

(c) The generalized momentum conjugate to θ is

$$p_{\theta} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}. \quad (2.11)$$

Since

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0,$$

it follows that

$$\boxed{\frac{dp_{\theta}}{dt} = 0.} \quad (2.12)$$

Hence

$$\boxed{L = mr^2\dot{\theta} = \text{constant},} \quad (2.13)$$

which is precisely the angular momentum of the particle about the origin.

(d) Since

$$L = mr^2\dot{\theta}, \quad (2.14)$$

we have

$$\dot{\theta} = \frac{L}{mr^2}. \quad (2.15)$$

Substituting this into the total energy,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) \quad (2.16)$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r). \quad (2.17)$$

Therefore,

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r), \quad (2.18)$$

where the effective potential is

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}. \quad (2.19)$$

The second term,

$$\frac{L^2}{2mr^2}, \quad (2.20)$$

is known as the centrifugal potential. It represents the effect of angular motion and prevents the particle from reaching the origin unless the angular momentum is zero. The radial motion can therefore be regarded as the motion of a one-dimensional particle in the effective potential $V_{\text{eff}}(r)$.

Problem 3

10 Marks

A particle of mass m is constrained to move on the surface of a sphere of radius R under the influence of gravity.

- (a) Write down the Lagrangian using suitable generalized coordinates.
- (b) Derive the equations of motion.
- (c) Identify all conserved quantities.

Solution

- (a) Let the particle be described by the spherical polar coordinates (r, θ, ϕ) . Since the particle is constrained to move on the surface of a sphere of radius R ,

$$r = R, \quad \dot{r} = 0.$$

The kinetic energy is therefore

$$T = \frac{1}{2}mR^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2). \quad (3.1)$$

Taking the downward vertical direction as the direction of gravity, the potential energy is

$$V = mgR \cos \theta. \quad (3.2)$$

Hence the Lagrangian is

$$\mathcal{L} = \frac{1}{2}mR^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - mgR \cos \theta. \quad (3.3)$$

(b) Applying the Euler–Lagrange equation to the coordinate θ ,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \quad (3.4)$$

we obtain

$$mR^2 \ddot{\theta} - mR^2 \sin \theta \cos \theta \dot{\phi}^2 - mgR \sin \theta = 0, \quad (3.5)$$

or

$$\ddot{\theta} = \sin \theta \cos \theta \dot{\phi}^2 + \frac{g}{R} \sin \theta. \quad (3.6)$$

For the coordinate ϕ ,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \quad (3.7)$$

Since the Lagrangian is independent of ϕ ,

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0,$$

and therefore

$$\frac{d}{dt} (mR^2 \sin^2 \theta \dot{\phi}) = 0. \quad (3.8)$$

(c) Since ϕ is a cyclic coordinate, its conjugate momentum is conserved:

$$p_\phi = mR^2 \sin^2 \theta \dot{\phi} = \text{constant}. \quad (3.9)$$

This is the component of the angular momentum about the vertical axis.

Furthermore, the Lagrangian has no explicit time dependence. Hence the total mechanical energy is conserved:

$$E = \dot{\theta} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} + \dot{\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \quad (3.10)$$

$$= \frac{1}{2} mR^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + mgR \cos \theta. \quad (3.11)$$

Therefore,

$$E = \frac{1}{2} mR^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + mgR \cos \theta = \text{constant}. \quad (3.12)$$

Thus, the conserved quantities are:

- the generalized momentum $p_\phi = mR^2 \sin^2 \theta \dot{\phi}$, corresponding to rotational symmetry about the vertical axis;
- the total mechanical energy E , corresponding to time-translation symmetry.

Problem 4

10 Marks

A simple pendulum of length ℓ and bob mass m oscillates freely.

- Obtain the Lagrangian.
- Derive the equation of motion.
- Linearize the equation for small oscillations.
- Determine the oscillation frequency.

Solution

- Let θ denote the angular displacement of the pendulum from the vertical.

The bob moves along a circular arc of radius ℓ , so its speed is

$$v = \ell \dot{\theta}. \quad (4.1)$$

Hence, the kinetic energy is

$$T = \frac{1}{2} m \ell^2 \dot{\theta}^2. \quad (4.2)$$

Choosing the lowest point of the pendulum as the zero of potential energy,

$$V = mg\ell(1 - \cos \theta). \quad (4.3)$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2} m \ell^2 \dot{\theta}^2 - mg\ell(1 - \cos \theta). \quad (4.4)$$

- Applying the Euler–Lagrange equation,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \quad (4.5)$$

we obtain

$$m\ell^2 \ddot{\theta} + mg\ell \sin \theta = 0. \quad (4.6)$$

Dividing throughout by $m\ell$ gives

$$\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0. \quad (4.7)$$

This is the exact nonlinear equation of motion of a simple pendulum.

(c) For small oscillations,

$$|\theta| \ll 1,$$

so that

$$\sin \theta \simeq \theta. \quad (4.8)$$

Hence, the equation of motion becomes

$$\ddot{\theta} + \frac{g}{\ell} \theta = 0, \quad (4.9)$$

which is the standard equation of simple harmonic motion.

(d) Comparing the above equation with

$$\ddot{x} + \omega^2 x = 0, \quad (4.10)$$

the angular frequency is

$$\omega = \sqrt{\frac{g}{\ell}}. \quad (4.11)$$

The corresponding time period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\ell}{g}}. \quad (4.12)$$

Thus, for small oscillations, the period is independent of the mass of the bob and depends only on the length of the pendulum and the acceleration due to gravity.

Problem 5

12 Marks

Two particles of masses m_1 and m_2 are connected by a light spring of spring constant k and move without friction along a horizontal line.

- (a) Write the Lagrangian of the system.
- (b) Introduce center-of-mass and relative coordinates.
- (c) Show that the motion separates into independent degrees of freedom.
- (d) Determine the normal-mode frequency.

Solution

- (a) Let x_1 and x_2 denote the positions of the two masses. Let the natural length of the spring be ℓ_0 .

The kinetic energy of the system is

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2. \quad (5.1)$$

The potential energy stored in the spring is

$$V = \frac{1}{2}k(x_2 - x_1 - \ell_0)^2. \quad (5.2)$$

Hence, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1 - \ell_0)^2. \quad (5.3)$$

(b) Introduce the center-of-mass coordinate

$$X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \quad (5.4)$$

and the relative coordinate

$$x = x_2 - x_1 - \ell_0. \quad (5.5)$$

Let

$$M = m_1 + m_2$$

be the total mass and

$$\mu = \frac{m_1m_2}{m_1 + m_2} \quad (5.6)$$

be the reduced mass.

The inverse transformations are

$$x_1 = X - \frac{m_2}{M}(x + \ell_0), \quad (5.7)$$

$$x_2 = X + \frac{m_1}{M}(x + \ell_0). \quad (5.8)$$

Differentiating,

$$\dot{x}_1 = \dot{X} - \frac{m_2}{M}\dot{x}, \quad (5.9)$$

$$\dot{x}_2 = \dot{X} + \frac{m_1}{M}\dot{x}. \quad (5.10)$$

(c) Substituting into the kinetic energy,

$$T = \frac{1}{2}m_1 \left(\dot{X} - \frac{m_2}{M}\dot{x} \right)^2 + \frac{1}{2}m_2 \left(\dot{X} + \frac{m_1}{M}\dot{x} \right)^2. \quad (5.11)$$

Expanding and simplifying,

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{x}^2. \quad (5.12)$$

Since

$$V = \frac{1}{2}kx^2,$$

the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}kx^2. \quad (5.13)$$

The center-of-mass coordinate satisfies

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{X}} \right) - \frac{\partial \mathcal{L}}{\partial X} = 0, \quad (5.14)$$

which gives

$$M\ddot{X} = 0. \quad (5.15)$$

Thus the center of mass moves with constant velocity.

For the relative coordinate,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0, \quad (5.16)$$

which gives

$$\mu\ddot{x} + kx = 0. \quad (5.17)$$

Hence the translational and internal motions are completely independent.

(d) The relative motion satisfies the equation of a simple harmonic oscillator,

$$\mu\ddot{x} + kx = 0. \quad (5.18)$$

Comparing with

$$\ddot{x} + \omega^2 x = 0, \quad (5.19)$$

we obtain the normal-mode angular frequency

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}. \quad (5.20)$$

The corresponding period of oscillation is

$$T = 2\pi\sqrt{\frac{\mu}{k}}. \quad (5.21)$$

Thus, the system consists of a freely moving center of mass together with an independent simple harmonic oscillation of the relative coordinate.

Problem 6

8 Marks

Consider a particle moving in the two-dimensional potential

$$V(x, y) = \frac{k}{2}(x^2 + y^2). \quad (6.1)$$

- (a) Write down the Hamiltonian.
- (b) Derive Hamilton's equations.
- (c) Verify that the total energy is conserved.

Solution

- (a) The kinetic energy of the particle is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \quad (6.2)$$

and the potential energy is

$$V = \frac{k}{2}(x^2 + y^2). \quad (6.3)$$

Hence, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2). \quad (6.4)$$

The generalized momenta are

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}, \quad (6.5)$$

$$p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}. \quad (6.6)$$

Therefore,

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}.$$

The Hamiltonian is

$$H = p_x\dot{x} + p_y\dot{y} - \mathcal{L} \quad (6.7)$$

$$= \frac{p_x^2 + p_y^2}{2m} + \frac{k}{2}(x^2 + y^2). \quad (6.8)$$

Hence,

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{k}{2} (x^2 + y^2). \quad (6.9)$$

(b) Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \quad (6.10)$$

For the x -coordinate,

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \quad (6.11)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -kx. \quad (6.12)$$

Similarly, for the y -coordinate,

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \quad (6.13)$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -ky. \quad (6.14)$$

Thus, Hamilton's equations are

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad (6.15)$$

and

$$\dot{p}_x = -kx, \quad \dot{p}_y = -ky. \quad (6.16)$$

Differentiating the first pair and using the second,

$$m\ddot{x} + kx = 0, \quad (6.17)$$

$$m\ddot{y} + ky = 0, \quad (6.18)$$

showing that the particle performs independent simple harmonic motion along the two coordinate axes.

(c) The total energy of the system is the Hamiltonian,

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + \frac{k}{2} (x^2 + y^2). \quad (6.19)$$

Differentiating with respect to time,

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y} + \frac{\partial H}{\partial p_x} \dot{p}_x + \frac{\partial H}{\partial p_y} \dot{p}_y. \quad (6.20)$$

Using Hamilton's equations,

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \frac{\partial H}{\partial p_x} + \frac{\partial H}{\partial y} \frac{\partial H}{\partial p_y} - \frac{\partial H}{\partial p_x} \frac{\partial H}{\partial x} - \frac{\partial H}{\partial p_y} \frac{\partial H}{\partial y} \quad (6.21)$$

$$= 0. \quad (6.22)$$

Hence,

$$\boxed{\frac{dH}{dt} = 0}, \quad (6.23)$$

which shows that the total energy is conserved.

Problem 7

10 Marks

A bead of mass m slides without friction on a circular wire of radius R rotating with constant angular velocity ω about its vertical diameter.

- (a) Obtain the Lagrangian.
- (b) Derive the equation of motion.
- (c) Find the equilibrium positions.
- (d) Determine the condition for their stability.

Solution

- (a) Let θ be the angular position of the bead measured from the downward vertical. The hoop rotates about its vertical diameter with a constant angular velocity ω .

The Cartesian coordinates of the bead are

$$x = R \sin \theta \cos \omega t, \quad (7.1)$$

$$y = R \sin \theta \sin \omega t, \quad (7.2)$$

$$z = -R \cos \theta. \quad (7.3)$$

Differentiating with respect to time, the speed of the bead is

$$v^2 = R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta. \quad (7.4)$$

Hence, the kinetic energy is

$$T = \frac{1}{2} m R^2 (\dot{\theta}^2 + \omega^2 \sin^2 \theta). \quad (7.5)$$

The gravitational potential energy is

$$V = -mgR \cos \theta. \quad (7.6)$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}mR^2 (\dot{\theta}^2 + \omega^2 \sin^2 \theta) + mgR \cos \theta. \quad (7.7)$$

(b) Applying the Euler–Lagrange equation,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \quad (7.8)$$

gives

$$mR^2 \ddot{\theta} - mR^2 \omega^2 \sin \theta \cos \theta + mgR \sin \theta = 0. \quad (7.9)$$

Hence,

$$\ddot{\theta} = \omega^2 \sin \theta \cos \theta - \frac{g}{R} \sin \theta. \quad (7.10)$$

(c) The equilibrium positions satisfy

$$\ddot{\theta} = 0.$$

Thus,

$$\sin \theta \left(\omega^2 \cos \theta - \frac{g}{R} \right) = 0. \quad (7.11)$$

Hence, the equilibrium positions are

$$\theta = 0, \quad \theta = \pi, \quad (7.12)$$

and, provided

$$\omega^2 \geq \frac{g}{R},$$

two additional equilibrium positions are obtained from

$$\cos \theta = \frac{g}{R\omega^2}. \quad (7.13)$$

(d) The effective potential is

$$V_{\text{eff}} = -mgR \cos \theta - \frac{1}{2}mR^2 \omega^2 \sin^2 \theta. \quad (7.14)$$

An equilibrium is stable if

$$\frac{d^2 V_{\text{eff}}}{d\theta^2} > 0.$$

Since

$$\frac{dV_{\text{eff}}}{d\theta} = mR \sin \theta (g - R\omega^2 \cos \theta), \quad (7.15)$$

we obtain

$$\frac{d^2 V_{\text{eff}}}{d\theta^2} = mR [g \cos \theta - R\omega^2 \cos 2\theta]. \quad (7.16)$$

Evaluating at the equilibrium positions,

- At $\theta = 0$,

$$\frac{d^2 V_{\text{eff}}}{d\theta^2} = mR(g - R\omega^2).$$

Hence,

$$\boxed{\theta = 0 \text{ is stable if } \omega < \sqrt{\frac{g}{R}}.} \quad (7.17)$$

- At $\theta = \pi$,

$$\frac{d^2 V_{\text{eff}}}{d\theta^2} = -mR(g + R\omega^2) < 0,$$

so

$$\boxed{\theta = \pi \text{ is always unstable.}} \quad (7.18)$$

- For

$$\cos \theta = \frac{g}{R\omega^2},$$

one finds

$$\boxed{\frac{d^2 V_{\text{eff}}}{d\theta^2} = mR^2 \omega^2 \sin^2 \theta > 0,} \quad (7.19)$$

so these two equilibrium positions are always stable whenever they exist, i.e.,

$$\boxed{\omega > \sqrt{\frac{g}{R}}.} \quad (7.20)$$

Thus, as the angular velocity passes through the critical value

$$\omega_c = \sqrt{\frac{g}{R}},$$

the stable equilibrium shifts from the bottom of the hoop to two symmetric positions on either side of the vertical.