Fundamentals of Signal Enhancement and Array Signal Processing
Solution Manual

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9 Differential Beamforming

9.2

Show that the coefficients of the Nth-order hypercardioid are given by

$$\mathbf{a}_{N,\max} = \frac{\mathbf{H}_N^{-1}\mathbf{1}}{\mathbf{1}^T\mathbf{H}_N^{-1}\mathbf{1}}.$$

Solution:
First we can notice that the rank of the matrix 11^T is 1. So , the rank of the matrix $H_N{}^{-1}11^T$ is 1 or 0. Therefor, for the non zero eigenvalue , the eigenvector is the maximum.
lets show that a_N is eigenvector for non zero eigenvalue:

$$\left(H_N{}^{-1}11^T\right)a_N = \left(H_N{}^{-1}11^T\right)\frac{H_N{}^{-1}1}{1^T H_N{}^{-1}1} = \frac{H_N{}^{-1}1}{1^T H_N{}^{-1}1}1H_N{}^{-1}1 = a_N\left(1H_N{}^{-1}1\right) = a_N \cdot \lambda$$

which $\lambda \triangleq 1H_N{}^{-1}1$ scalar , non zero eigenvalue of $H_N{}^{-1}11^T$
a_N is the eigenvector corresponding to the maximum eigenvalue of the matrix.

$\blacksquare$

9.3

Using the definition of the frequency-independent FBR of a theoretical Nth-order DSA (**??**), show that

$$\mathcal{F}(\mathbf{a}_N) = \frac{\mathbf{a}_N^T\mathbf{H}_N''\mathbf{a}_N}{\mathbf{a}_N^T\mathbf{H}_N'\mathbf{a}_N},$$

where $\mathbf{H}_N'$ and $\mathbf{H}_N''$ are Hankel matrices.
Solution:
We know that the FBR define as:

$$F(a_N) = \frac{\int_0^{\frac{\pi}{2}} B^2(a_N,\cos\theta)\sin\theta d\theta}{\int_{\frac{\pi}{2}}^{\pi} B^2(a_N,\cos\theta)\sin\theta d\theta}$$

and B define as:

$$B(a_N,\cos\theta) = \sum_{n=0}^{n=N} a_{N,n}\cos^n\theta$$

now we substitute B in FBR definition:

$$F(a_N) = \frac{\int_0^{\frac{\pi}{2}} \sum_{n=0}^{n=N} a_{N,n}\cos^n\theta \sum_{k=0}^{k=N} a_{N,k}\cos^k\theta\sin\theta d\theta}{\int_{\frac{\pi}{2}}^{\pi} \sum_{n=0}^{n=N} a_{N,n}\cos^n\theta \sum_{k=0}^{k=N} a_{N,k}\cos^k\theta\sin\theta d\theta} = \frac{\sum_{k=0}^{k=N}\sum_{n=0}^{n=N} a_{N,n}a_{N,k}\int_0^{\frac{\pi}{2}}\cos^n\theta\cos^k\theta\sin\theta d\theta}{\sum_{k=0}^{k=N}\sum_{n=0}^{n=N} a_{N,n}a_{N,k}\int_{\frac{\pi}{2}}^{\pi}\cos^n\theta\cos^k\theta\sin\theta d\theta} = \otimes$$

using the following solutions:

$$\int_0^{\frac{\pi}{2}} \cos^i\theta \cos^j\theta \sin\theta d\theta = \frac{1}{1+i+j} = [H''_N]_{ij}$$

$$\int_{\frac{\pi}{2}}^{\pi} \cos^i\theta \cos^j\theta \sin\theta d\theta = \frac{(-1)^{i+j}}{1+i+j} = [H'_N]_{ij}$$

finally:

$$\otimes = \frac{\sum_{k=0}^{k=N}\sum_{n=0}^{n=N} a_{N,n}a_{N,k}[H''_N]_{nk}}{\sum_{k=0}^{k=N}\sum_{n=0}^{n=N} a_{N,n}a_{N,k}[H'_N]_{nk}} = \frac{a_N{}^T H''_N a_N}{a_N{}^T H'_N a_N}$$

where H''_N and H'_N are two Hankel matrices.

■

9.5

Show that the directivity pattern of the first-order hypercardioid can be expressed as

$$\mathcal{B}_{1,\mathrm{Hd}}(\cos\theta) = \frac{1}{4} + \frac{3}{4}\cos\theta.$$

Solution:
We know the definition of B:

$$B_{N,Hd}(\cos\theta) = \frac{1^T H^{-1}{}_N p(\cos\theta)}{1^T H^{-1}{}_N 1}$$

and for N=1:

$$p(\cos\theta) = \begin{bmatrix}1 & \cos\theta\end{bmatrix}^T$$
$$1^T = \begin{bmatrix}1 & 1\end{bmatrix}$$

$$1 = \begin{bmatrix}1 & 1\end{bmatrix}^T$$

$$H_1 = \begin{pmatrix}1 & 0\\ 0 & \frac{1}{3}\end{pmatrix} \to H_1{}^{-1} = \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$$

substituting:

$$B_{N,Hd}(\cos\theta) = \frac{\begin{bmatrix}1 & 1\end{bmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\begin{bmatrix}1 & \cos\theta\end{bmatrix}^T}{\begin{bmatrix}1 & 1\end{bmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\begin{bmatrix}1 & 1\end{bmatrix}^T} = \frac{1+3\cos\theta}{4} = \frac{1}{4} + \frac{3\cos\theta}{4}$$

■

9.6

Show that the directivity pattern of the first-order supercardioid can be expressed as

$$\mathcal{B}_{1,\mathrm{Sd}}(\cos\theta) = \frac{\sqrt{3}-1}{2} + \frac{3-\sqrt{3}}{2}\cos\theta.$$

Solution:
The beampattern of the Nth-order supercardioid is

$$B_{N,Sd}(\cos\theta) = \frac{a'^T_{N,\max} p(\cos\theta)}{a'^T_{N,\max} p(1)}$$

when $a'^T_{N,\max}$ is the eigenvector corresponding to the maximum eigenvalue of $H'^{-1}_N H_1''$

for N=1:

$$H_1' = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{3} \end{pmatrix} \rightarrow H_1'^{-1} = \begin{pmatrix} 4 & 6 \\ 6 & 12 \end{pmatrix}$$

$$H_1'' = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

$$H'^{-1}_N H_1'' = \begin{pmatrix} 4 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 12 & 7 \end{pmatrix}$$

the max eigenvalue of the matrix $\begin{pmatrix} 7 & 4 \\ 12 & 7 \end{pmatrix}$ is $\lambda_{\max} = 7 + 4\sqrt{3}$ the eigenvector for the max eigenvalue is : $\begin{pmatrix} \sqrt{3}-1 \\ 3-\sqrt{3} \end{pmatrix}$
So:

$$a'_{N,\max} = \begin{pmatrix} \sqrt{3}-1 \\ 3-\sqrt{3} \end{pmatrix}$$

And finally:

$$B_{1,Sd}(\cos\theta) = \frac{\begin{pmatrix} \sqrt{3}-1 & 3-\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & \cos\theta \end{pmatrix}^T}{\sqrt{3}-1+3-\sqrt{3}} = \frac{\sqrt{3}-1}{2} + \frac{3-\sqrt{3}}{2}\cos\theta$$

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9.7

Show that the directivity pattern of the second-order hypercardioid can be expressed as

$$\mathcal{B}_{2,\mathrm{Hd}}(\cos\theta) = -\frac{1}{6} + \frac{1}{3}\cos\theta + \frac{5}{6}\cos^2\theta.$$

Solution:
We know the definition of B:

$$B_{N,Hd}(\cos\theta) = \frac{1^T H^{-1}{}_N p(\cos\theta)}{1^T H^{-1}{}_N 1}$$

and for N=2:

$$p(\cos\theta) = \begin{bmatrix} 1 & \cos\theta & \cos^2\theta \end{bmatrix}^T$$

$$1^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$$

$$H_1 = \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} \end{pmatrix} \rightarrow H_1^{-1} = \begin{pmatrix} \frac{9}{4} & 0 & \frac{-15}{4} \\ 0 & \frac{12}{4} & 0 \\ \frac{-15}{4} & 0 & \frac{45}{4} \end{pmatrix}$$

and now:

$$B_{2,Hd} = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} \frac{9}{4} & 0 & \frac{-15}{4} \\ 0 & \frac{12}{4} & 0 \\ \frac{-15}{4} & 0 & \frac{45}{4} \end{pmatrix} \begin{bmatrix} 1 & \cos\theta & \cos^2\theta \end{bmatrix}^T}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} \frac{9}{4} & 0 & \frac{-15}{4} \\ 0 & \frac{12}{4} & 0 \\ \frac{-15}{4} & 0 & \frac{45}{4} \end{pmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T} = \frac{\begin{bmatrix} -\frac{6}{4} & \frac{12}{3} & \frac{30}{4} \end{bmatrix} \begin{bmatrix} 1 & \cos\theta & \cos^2\theta \end{bmatrix}^T}{\frac{36}{4}} =$$

$$= -\frac{1}{6} + \frac{1}{3}\cos\theta + \frac{5}{6}\cos^2\theta$$

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9.8

Show that the directivity pattern of the second-order supercardioid can be expressed as

$$\mathcal{B}_{2,\mathrm{Sd}}(\cos\theta) = \frac{1}{2\left(3+\sqrt{7}\right)} + \frac{\sqrt{7}}{3+\sqrt{7}}\cos\theta + \frac{5}{2\left(3+\sqrt{7}\right)}\cos^2\theta.$$

Solution:
The beampattern of the Nth-order supercardioid is

$$B_{N,Sd}(\cos\theta) = \frac{a_{N,\max}^{'T} p(\cos\theta)}{a_{N,\max}^{'T} p(1)}$$

when $a_{N,\max}^{'T}$ is the eigenvector corresponding to the maximum eigenvalue of $H_N^{'-1} H_1''$

for N=2:

$$H_2' = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4} \\ \frac{1}{3} & -\frac{1}{4} & \frac{1}{5} \end{pmatrix} \rightarrow H_2\prime^{-1} = \begin{pmatrix} 9 & 36 & 30 \\ 36 & 192 & 180 \\ 30 & 180 & 180 \end{pmatrix}$$

$$H_2'' = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

$$H_2^{'-1} H_2'' = \begin{pmatrix} 9 & 36 & 30 \\ 36 & 192 & 180 \\ 30 & 180 & 180 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 37 & 24 & 18 \\ 192 & 127 & 96 \\ 180 & 120 & 91 \end{pmatrix}$$

the max eigenvalue of the matrix $H_2^{'-1} H_2''$ is $\lambda_{\max} = 127 + 48\sqrt{7}$ the eigenvector for the max eigenvalue is : $\begin{pmatrix} 1 \\ 2\sqrt{7} \\ 5 \end{pmatrix}$

So:

$$a_{2,\max}' = \begin{pmatrix} 1 \\ 2\sqrt{7} \\ 5 \end{pmatrix}$$

And finally:

$$B_{2,Sd}(\cos\theta) = \frac{\begin{pmatrix} 1 \\ 2\sqrt{7} \\ 5 \end{pmatrix}^T \begin{pmatrix} 1 & \cos\theta & \cos^2\theta \end{pmatrix}^T}{1 + 2\sqrt{7} + 5} = \frac{1}{2(3+\sqrt{7})} + \frac{2\sqrt{7}}{2(3+\sqrt{7})}\cos\theta + \frac{5}{2(3+\sqrt{7})}\cos^2\theta$$

$\blacksquare$

9.9

Show that the directivity pattern of the third-order hypercardioid can be expressed as

$$\mathcal{B}_{3,\mathrm{Hd}}(\cos\theta) = -\frac{3}{32} - \frac{15}{32}\cos\theta + \frac{15}{32}\cos^2\theta + \frac{35}{32}\cos^3\theta.$$

Solution:
We know the definition of B:

$$B_{N,Hd}(\cos\theta) = \frac{1^T {H^{-1}}_N p(\cos\theta)}{1^T {H^{-1}}_N 1}$$

and for N=3:

$$p(\cos\theta) = \begin{bmatrix} 1 & \cos\theta & \cos^2\theta & \cos^3\theta \end{bmatrix}^T$$

$$1^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$1 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$$

$$H_1 = \begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{7} \end{pmatrix} \rightarrow {H_1}^{-1} = \begin{pmatrix} \frac{9}{4} & 0 & \frac{-15}{4} & 0 \\ 0 & \frac{75}{4} & 0 & \frac{-105}{4} \\ \frac{-15}{4} & 0 & \frac{45}{4} & 0 \\ 0 & \frac{-105}{4} & 0 & \frac{175}{4} \end{pmatrix}$$

substituting:

$$B_{3,Hd} = \frac{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \begin{pmatrix} \frac{9}{4} & 0 & \frac{-15}{4} & 0 \\ 0 & \frac{75}{4} & 0 & \frac{-105}{4} \\ \frac{-15}{4} & 0 & \frac{45}{4} & 0 \\ 0 & \frac{-105}{4} & 0 & \frac{175}{4} \end{pmatrix} \begin{bmatrix}1 & \cos\theta & \cos^2\theta & \cos^3\theta\end{bmatrix}^T}{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \begin{pmatrix} \frac{9}{4} & 0 & \frac{-15}{4} & 0 \\ 0 & \frac{75}{4} & 0 & \frac{-105}{4} \\ \frac{-15}{4} & 0 & \frac{45}{4} & 0 \\ 0 & \frac{-105}{4} & 0 & \frac{175}{4} \end{pmatrix} \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T} =$$

$$= \frac{-\frac{3}{2} - \frac{15}{2}\cos\theta + \frac{15}{2}\cos^2\theta + \frac{35}{2}\cos^3\theta}{16} = -\frac{3}{32} - \frac{15}{32}\cos\theta + \frac{15}{32}\cos^2\theta + \frac{35}{32}\cos^3\theta$$

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9.11

Show that the beampattern, the DF, and the WNG of the first-order DSA can be approximated as

$$\mathcal{B}\left[\mathbf{h}_1(f), \cos\theta\right] \approx \frac{1}{1-\alpha_{1,1}}\cos\theta - \frac{\alpha_{1,1}}{1-\alpha_{1,1}},$$

$$\mathcal{D}\left[\mathbf{h}_1(f)\right] \approx \frac{\left(1-\alpha_{1,1}\right)^2}{\alpha_{1,1}^2 + \frac{1}{3}},$$

$$\mathcal{W}\left[\mathbf{h}_1(f)\right] \approx \frac{1}{2}\left[2\pi f\tau_0\left(1-\alpha_{1,1}\right)\right]^2.$$

Solution:
First we know that the beampattern is:

$$B[h_1(f), \cos\theta] = \frac{1 - e^{j2\pi f\tau_0(\cos\theta - \alpha_{1,1})}}{1 - e^{j2\pi f\tau_0(1-\alpha_{1,1})}}$$

using the approximation:

$$e^x \approx 1 + x$$

we can approximate the beampattern as:

$$B[h_1(f), \cos\theta] = \frac{1 - e^{j2\pi f\tau_0(\cos\theta - \alpha_{1,1})}}{1 - e^{j2\pi f\tau_0(1-\alpha_{1,1})}} \approx \frac{1 - 1 - j2\pi f\tau_0(\cos\theta - \alpha_{1,1})}{1 - 1 - j2\pi f\tau_0(1-\alpha_{1,1})} = \frac{1}{1-\alpha_{1,1}}\cos\theta - \frac{\alpha_{1,1}}{1-\alpha_{1,1}}$$

■

we know that the DF is:

$$D[h_1(f)] = \frac{1 - \cos[2\pi f\tau_0(1-\alpha_{1,1})]}{1 - \sin c(2\pi f\tau_0)\cos(2\pi f\tau_0\alpha_{1.1})}$$

Using the approximations:

$$\cos x \approx 1 - \frac{x^2}{2}$$

$$\sin cx \approx 1 - \frac{x^2}{6}$$

the DF becomes:

$$D[h_1(f)] \approx \frac{1 - 1 + \frac{[2\pi f\tau_0(1-\alpha_{1,1})]^2}{2}}{1 - \left(1 - \frac{(2\pi f\tau_0)^2}{6}\right)\left(1 - \frac{(2\pi f\tau_0\alpha_{1.1})^2}{2}\right)} = \frac{\frac{[2\pi f\tau_0(1-\alpha_{1,1})]^2}{2}}{1 - \left(1 - \frac{(2\pi f\tau_0)^2}{6} - \frac{(2\pi f\tau_0\alpha_{1.1})^2}{2} + \frac{(2\pi f\tau_0)^4\alpha_{1.1}{}^2}{12}\right)}$$

$$\approx \frac{\frac{[2\pi f\tau_0(1-\alpha_{1,1})]^2}{2}}{\frac{(2\pi f\tau_0)^2}{6}+\frac{(2\pi f\tau_0\alpha_{1.1})^2}{2}} = \frac{(1-\alpha_{1,1})^2}{\frac{1}{3}+\alpha_{1.1}{}^2}$$

$\blacksquare$

The WNG is:

$$W[h_1(f)] = \frac{1}{2}\left|1-e^{j2\pi f\tau_0(1-\alpha_{1,1})}\right|^2 = 1-\cos[2\pi f\tau_0(1-\alpha_{1,1})]$$

using the cos approximation:

$$W[h_1(f)] \approx 1-1+\frac{[2\pi f\tau_0(1-\alpha_{1,1})]^2}{2} = \frac{1}{2}[2\pi f\tau_0(1-\alpha_{1,1})]$$

$\blacksquare$

9.12

Show that the inverse of the Vandermonde matrix $\mathbf{V}(f)$ that appears in (**??**) is given by

$$\mathbf{V}^{-1}(f) =$$
$$\begin{bmatrix} \frac{v_2v_3}{(v_2-v_1)(v_3-v_1)} & -\frac{v_1v_3}{(v_2-v_1)(v_3-v_2)} & \frac{v_1v_2}{(v_3-v_1)(v_3-v_2)} \\ -\frac{v_2+v_3}{(v_2-v_1)(v_3-v_1)} & \frac{v_1+v_3}{(v_2-v_1)(v_3-v_2)} & -\frac{v_1+v_2}{(v_3-v_1)(v_3-v_2)} \\ \frac{1}{(v_2-v_1)(v_3-v_1)} & -\frac{1}{(v_2-v_1)(v_3-v_2)} & \frac{1}{(v_3-v_1)(v_3-v_2)} \end{bmatrix}.$$

Solution:

The inverse of the Vandermonde matrix is given by:

$$V^{-1}(f) = U(f)L(f)$$

where,

$$U(f) = \begin{bmatrix} 1 & -v_1 & v_1v_2 \\ 0 & 1 & -(v_1+v_2) \\ 0 & 0 & 1 \end{bmatrix}$$

$$L(f) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{v_1-v_2} & \frac{1}{v_2-v_1} & 0 \\ \frac{1}{(v_1-v_2)(v_1-v_3)} & \frac{1}{(v_2-v_1)(v_2-v_3)} & \frac{1}{(v_3-v_1)(v_3-v_2)} \end{bmatrix}$$

substituting U L in order to find the inverse of the Vandermonde matrix :

$$V^{-1}(f) = \begin{bmatrix} 1 & -v_1 & v_1v_2 \\ 0 & 1 & -(v_1+v_2) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{v_1-v_2} & \frac{1}{v_2-v_1} & 0 \\ \frac{1}{(v_1-v_2)(v_1-v_3)} & \frac{1}{(v_2-v_1)(v_2-v_3)} & \frac{1}{(v_3-v_1)(v_3-v_2)} \end{bmatrix} =$$

$$= \begin{bmatrix} 1-\frac{v_1}{v_1-v_2}+\frac{v_1v_2}{(v_1-v_2)(v_1-v_3)} & -\frac{v_1}{v_2-v_1}+\frac{v_1v_2}{(v_2-v_1)(v_2-v_3)} & \frac{v_1v_2}{(v_3-v_1)(v_3-v_2)} \\ \frac{1}{v_1-v_2}-\frac{(v_1+v_2)}{(v_1-v_2)(v_1-v_3)} & \frac{1}{v_2-v_1}-\frac{(v_1+v_2)}{(v_2-v_1)(v_2-v_3)} & \frac{-(v_1+v_2)}{(v_3-v_1)(v_3-v_2)} \\ \frac{1}{(v_1-v_2)(v_1-v_3)} & \frac{1}{(v_2-v_1)(v_2-v_3)} & \frac{1}{(v_3-v_1)(v_3-v_2)} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{(v_1-v_2)(v_1-v_3)-v_1(v_1-v_3)+v_1v_2}{(v_1-v_2)(v_1-v_3)} & \frac{-v_1(v_2-v_3)+v_1v_2}{(v_2-v_1)(v_2-v_3)} & \frac{v_1v_2}{(v_3-v_1)(v_3-v_2)} \\ \frac{(v_1-v_3)-(v_1+v_2)}{(v_1-v_2)(v_1-v_3)} & \frac{(v_2-v_3)-(v_1+v_2)}{(v_2-v_1)(v_2-v_3)} & \frac{-(v_1+v_2)}{(v_3-v_1)(v_3-v_2)} \\ \frac{1}{(v_1-v_2)(v_1-v_3)} & \frac{1}{(v_2-v_1)(v_2-v_3)} & \frac{1}{(v_3-v_1)(v_3-v_2)} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{v_1v_1-v_1v_3-v_2v_1+v_2v_3-v_1v_1+v_1v_3+v_1v_2}{(v_1-v_2)(v_1-v_3)} & \frac{-v_1v_2+v_1v_3+v_1v_2}{(v_2-v_1)(v_2-v_3)} & \frac{v_1v_2}{(v_3-v_1)(v_3-v_2)} \\ \frac{v_1-v_3-v_1-v_2}{(v_1-v_2)(v_1-v_3)} & \frac{v_2-v_3-v_1-v_2}{(v_2-v_1)(v_2-v_3)} & \frac{-(v_1+v_2)}{(v_3-v_1)(v_3-v_2)} \\ \frac{1}{(v_1-v_2)(v_1-v_3)} & \frac{-1}{(v_2-v_1)(v_3-v_2)} & \frac{1}{(v_3-v_1)(v_3-v_2)} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{v_2v_3}{(v_2-v_1)(v_3-v_1)} & \frac{-v_1v_3}{(v_2-v_1)(v_3-v_2)} & \frac{v_1v_2}{(v_3-v_1)(v_3-v_2)} \\ \frac{-(v_3+v_2)}{(v_2-v_1)(v_3-v_1)} & \frac{v_1+v_3}{(v_2-v_1)(v_3-v_2)} & \frac{-(v_1+v_2)}{(v_3-v_1)(v_3-v_2)} \\ \frac{1}{(v_2-v_1)(v_3-v_1)} & \frac{-1}{(v_2-v_1)(v_3-v_2)} & \frac{1}{(v_3-v_1)(v_3-v_2)} \end{bmatrix}$$

$\blacksquare$

9.13

Show that in the case of a second-order DSA with a zero of multiplicity 2 in the beampattern, the beamformer is given by

$$\mathbf{h}_{2,0}(f) = \frac{1}{\left[1 - e^{\jmath 2\pi f \tau_0 (1-\alpha_{2,1})}\right]^2} \begin{bmatrix} 1 \\ -2e^{-\jmath 2\pi f \tau_0 \alpha_{2,1}} \\ e^{-\jmath 4\pi f \tau_0 \alpha_{2,1}} \end{bmatrix},$$

we want to solve the following linear problem:

$$\begin{bmatrix} d^H(f,1) \\ d^H(f,\alpha_{2,1}) \\ \sum d(f,\alpha_{2,1})^H \end{bmatrix} h(f) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & v_1 & {v_1}^2 \\ 1 & v_2 & {v_2}^2 \\ \multicolumn{3}{c}{\sum d(f,\alpha_{2,1})^H} \end{bmatrix} h(f) = \begin{bmatrix} 1 & e^{j2\pi f\tau_0} & e^{j4\pi f\tau_0} \\ 1 & e^{j2\pi f\tau_0 \alpha_{2,1}} & e^{j4\pi f\tau_0 \alpha_{2,1}} \\ \multicolumn{3}{c}{\sum d(f,\alpha_{2,1})^H} \end{bmatrix} h(f) = \begin{bmatrix} \begin{bmatrix} 1 & e^{j2\pi f\tau_0} & e^{j4\pi f\tau_0} \end{bmatrix} h(f) \\ \begin{bmatrix} 1 & e^{j2\pi f\tau_0 \alpha_{2,1}} & e^{j4\pi f\tau_0 \alpha_{2,1}} \end{bmatrix} h(f) \\ \sum d(f,\alpha_{2,1})^H h(f) \end{bmatrix} = \otimes$$

using that $\sum d(f,\alpha_{2,1})^H h(f) = 0$:

$$\rightarrow \otimes = \begin{bmatrix} \begin{bmatrix} 1 & e^{j2\pi f\tau_0} & e^{j4\pi f\tau_0} \end{bmatrix} h(f) \\ \begin{bmatrix} 1 & e^{j2\pi f\tau_0 \alpha_{2,1}} & e^{j4\pi f\tau_0 \alpha_{2,1}} \end{bmatrix} h(f) \\ 0 \end{bmatrix}$$

and finally its easy to see that the solution h(f) is:

$$h_{2,0}(f) = \frac{1}{\left[1 - e^{j2\pi f\tau_0(1-\alpha_{2,1})}\right]^2} \begin{bmatrix} 1 \\ -2e^{-j2\pi f\tau_0 \alpha_{2,1}} \\ e^{-j4\pi f\tau_0 \alpha_{2,1}} \end{bmatrix}$$

■

the beampattern can be written as

$$\mathcal{B}\left[\mathbf{h}_{2,0}(f), \cos\theta\right] \approx \frac{1}{(1-\alpha_{2,1})^2} (\cos\theta - \alpha_{2,1})^2,$$

Solution:
the beampattern has the form:

$$B[h_{2,0}(f), \cos\theta] = \frac{\left[1 - e^{j2\pi f\tau_0(\cos\theta - \alpha_{2,1})}\right]^2}{\left[1 - e^{j2\pi f\tau_0(1-\alpha_{2,1})}\right]^2}$$

using the following approximation:

$$e^x \approx 1 + x$$

we get:

$$B[h_{2,0}(f), \cos\theta] \approx \frac{[1 - 1 - j2\pi f\tau_0(\cos\theta - \alpha_{2,1})]^2}{[1 - 1 - j2\pi f\tau_0(1 - \alpha_{2,1})]^2} = \frac{[j2\pi f\tau_0(\cos\theta - \alpha_{2,1})]^2}{[j2\pi f\tau_0(1-\alpha_{2,1})]^2} = \frac{[\cos\theta - \alpha_{2,1}]^2}{[1-\alpha_{2,1}]^2}$$

■

The WNG can be approximated as

$$\mathcal{W}\left[\mathbf{h}_{2,0}(f)\right] \approx \frac{1}{6}\left[2\pi f\tau_0\left(1-\alpha_{2,1}\right)\right]^4.$$

Solution:
We find that the WNG is:

$$W[h(f)] = \frac{1}{6}\left|1 - e^{j2\pi f\tau_0(1-\alpha_{2,1})}\right|^4 = \frac{2}{3}[1 - \cos[2\pi f\tau_0(1-\alpha_{2,1})]]^2$$

with the following approximation:

$$\cos x \approx 1 - \frac{x^2}{2}$$

The WNG can be approximated as

$$W[h(f)] \approx \frac{2}{3}\left[1-1+\frac{[2\pi f\tau_0(1-\alpha_{2,1})]^2}{2}\right]^2 = \frac{2}{3\cdot 4}[2\pi f\tau_0(1-\alpha_{2,1})]^4 = \frac{1}{6}[2\pi f\tau_0(1-\alpha_{2,1})]^4$$

$\blacksquare$

9.14

Show that the first column of the inverse of the Vandermonde matrix $\mathbf{V}(f)$ that appears in (**??**) is given by

$$\mathbf{V}^{-1}(f;:,1) = \begin{bmatrix} \frac{v_2v_3v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} \\ -\frac{v_2v_3+v_3v_4+v_2v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} \\ \frac{v_2+v_3+v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} \\ -\frac{1}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} \end{bmatrix}.$$

<u>Solution:</u>
The inverse of the Vandermonde matrix is given by:

$$V^{-1}(f) = U(f)L(f)$$

where,

$$U(f) = \begin{pmatrix} 1 & -v_1 & v_1v_2 & -v_1v_2v_3 \\ 0 & 1 & -(v_1+v_2) & v_1v_2+v_1v_3+v_3v_2 \\ 0 & 0 & 1 & -(v_1+v_2+v_3) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L(f) = \begin{bmatrix} 1 & \cdots \\ \frac{1}{(v_1-v_2)} & \cdots \\ \frac{1}{(v_1-v_2)(v_1-v_3)} & \cdots \\ \frac{1}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \end{bmatrix}$$

substituting U L in order to find the inverse of the Vandermonde matrix :

$$V^{-1}(f) = U(f)L(f) = \begin{pmatrix} 1 & -v_1 & v_1v_2 & -v_1v_2v_3 \\ 0 & 1 & -(v_1+v_2) & v_1v_2+v_1v_3+v_3v_2 \\ 0 & 0 & 1 & -(v_1+v_2+v_3) \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 & \cdots \\ \frac{1}{(v_1-v_2)} & \cdots \\ \frac{1}{(v_1-v_2)(v_1-v_3)} & \cdots \\ \frac{1}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \end{bmatrix} =$$

$$= \begin{bmatrix} 1+\frac{-v_1}{(v_1-v_2)}+\frac{v_1v_2}{(v_1-v_2)(v_1-v_3)}+\frac{-v_1v_2v_3}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \\ \frac{1}{(v_1-v_2)}+\frac{-(v_1+v_2)}{(v_1-v_2)(v_1-v_3)}+\frac{v_1v_2+v_1v_3+v_3v_2}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \\ \frac{1}{(v_1-v_2)(v_1-v_3)}+\frac{-(v_1+v_2+v_3)}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \\ 1+\frac{1}{(v_1-v_2)}+\frac{1}{(v_1-v_2)(v_1-v_3)}+\frac{1}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \end{bmatrix} = \begin{bmatrix} \frac{-v_2v_3v_4}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \\ \frac{v_2v_3+v_3v_4+v_2v_4}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \\ \frac{-v_2-v_3-v_4}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \\ \frac{1}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} & \cdots \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{v_2v_3v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} & \cdots \\ -\frac{v_2v_3+v_3v_4+v_2v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} & \cdots \\ \frac{v_2+v_3+v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} & \cdots \\ -\frac{1}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} & \cdots \end{bmatrix}$$

so we get:

$$V^{-1}(f) = \begin{bmatrix} \frac{v_2v_3v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} & \cdots \\ -\frac{v_2v_3+v_3v_4+v_2v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} & \cdots \\ \frac{v_2+v_3+v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} & \cdots \\ -\frac{1}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} & \cdots \end{bmatrix}$$

$\blacksquare$

9.15

Show that the third-order DSA beamformer is given by

$$\mathbf{h}_3(f) = \frac{1}{\left[1-e^{\jmath 2\pi f\tau_0(1-\alpha_{3,1})}\right]\left[1-e^{\jmath 2\pi f\tau_0(1-\alpha_{3,2})}\right]\left[1-e^{\jmath 2\pi f\tau_0(1-\alpha_{3,3})}\right]} \times$$
$$\begin{bmatrix} 1 \\ -e^{-\jmath 2\pi f\tau_0\alpha_{3,1}} - e^{-\jmath 2\pi f\tau_0\alpha_{3,2}} - e^{-\jmath 2\pi f\tau_0\alpha_{3,3}} \\ e^{-\jmath 2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,2})} + e^{-\jmath 2\pi f\tau_0(\alpha_{3,2}+\alpha_{3,3})} + e^{-\jmath 2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,3})} \\ -e^{-\jmath 2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,2}+\alpha_{3,3})} \end{bmatrix}.$$

Solution:
first we know from (9.82):

$$V(f)h_3(f) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$h_3(f) = V^{-1}(f)\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = V^{-1}(f;:,1)$$

as we know from problem 9.14 :

$$\rightarrow h_3(f) = \begin{bmatrix} \frac{v_2v_3v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} \\ -\frac{v_2v_3+v_3v_4+v_2v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} \\ \frac{v_2+v_3+v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} \\ -\frac{1}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} \end{bmatrix}$$

Now let's simplify the phrase:

$$h_3(f,1) = \frac{v_2v_3v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} = \frac{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}{\left(e^{j2\pi f\tau_0\alpha_{3,1}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,2}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,3}}-e^{j2\pi f\tau_0}\right)} =$$

$$= \frac{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}{\left(e^{j2\pi f\tau_0\alpha_{3,1}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,2}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,3}}-e^{j2\pi f\tau_0}\right)}\frac{\frac{1}{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}}{\frac{1}{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}}$$

$$\rightarrow h_3(f,1) = \frac{1}{\left(1-e^{j2\pi f\tau_0(1-\alpha_{3,1})}\right)\left(1-e^{j2\pi f\tau_0(1-\alpha_{3,2})}\right)\left(1-e^{j2\pi f\tau_0(1-\alpha_{3,3})}\right)}$$

$$h_3(f,2) = -\frac{v_2v_3+v_3v_4+v_2v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} = -\frac{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}+e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}+e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,3}}}{\left(e^{j2\pi f\tau_0\alpha_{3,1}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,2}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,3}}-e^{j2\pi f\tau_0}\right)} =$$

$$= -\frac{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}+e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}+e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,3}}}{\left(e^{j2\pi f\tau_0\alpha_{3,1}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,2}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,3}}-e^{j2\pi f\tau_0}\right)}\frac{\frac{1}{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}}{\frac{1}{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}}$$

$$\rightarrow h_3(f,2) = -\frac{e^{-j2\pi f\tau_0\alpha_{3,1}}+e^{-j2\pi f\tau_0\alpha_{3,2}}+e^{-j2\pi f\tau_0\alpha_{3,3}}}{\left(1-e^{j2\pi f\tau_0(1-\alpha_{3,1})}\right)\left(1-e^{j2\pi f\tau_0(1-\alpha_{3,2})}\right)\left(1-e^{j2\pi f\tau_0(1-\alpha_{3,3})}\right)}$$

$$h_3(f,3) = \frac{v_2+v_3+v_4}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} = \frac{e^{j2\pi f\tau_0\alpha_{3,1}}+e^{j2\pi f\tau_0\alpha_{3,2}}+e^{j2\pi f\tau_0\alpha_{3,3}}}{\left(e^{j2\pi f\tau_0\alpha_{3,1}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,2}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,3}}-e^{j2\pi f\tau_0}\right)} =$$

$$= \frac{e^{j2\pi f\tau_0\alpha_{3,1}}+e^{j2\pi f\tau_0\alpha_{3,2}}+e^{j2\pi f\tau_0\alpha_{3,3}}}{\left(e^{j2\pi f\tau_0\alpha_{3,1}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,2}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,3}}-e^{j2\pi f\tau_0}\right)}\frac{\frac{1}{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}}{\frac{1}{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}}$$

$$\rightarrow h_3(f,3) = \frac{e^{-j2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,2})}+e^{-j2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,3})}+e^{-j2\pi f\tau_0(\alpha_{3,2}+\alpha_{3,3})}}{\left(1-e^{j2\pi f\tau_0(1-\alpha_{3,1})}\right)\left(1-e^{j2\pi f\tau_0(1-\alpha_{3,2})}\right)\left(1-e^{j2\pi f\tau_0(1-\alpha_{3,3})}\right)}$$

$$h_3(f,4) = -\frac{1}{(v_2-v_1)(v_3-v_1)(v_4-v_1)} = -\frac{1}{\left(e^{j2\pi f\tau_0\alpha_{3,1}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,2}}-e^{j2\pi f\tau_0}\right)\left(e^{j2\pi f\tau_0\alpha_{3,3}}-e^{j2\pi f\tau_0}\right)} =$$

$$= -\frac{1}{(e^{j2\pi f\tau_0\alpha_{3,1}} - e^{j2\pi f\tau_0})(e^{j2\pi f\tau_0\alpha_{3,2}} - e^{j2\pi f\tau_0})(e^{j2\pi f\tau_0\alpha_{3,3}} - e^{j2\pi f\tau_0})} \frac{\frac{1}{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}}{\frac{1}{e^{j2\pi f\tau_0\alpha_{3,1}}e^{j2\pi f\tau_0\alpha_{3,2}}e^{j2\pi f\tau_0\alpha_{3,3}}}}$$

$$\to h_3(f,4) = -\frac{e^{-j2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,2}+\alpha_{3,3})}}{(1-e^{j2\pi f\tau_0(1-\alpha_{3,1})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,2})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,3})})}$$

and finally:

$$h_3(f) = \begin{bmatrix} \frac{1}{(1-e^{j2\pi f\tau_0(1-\alpha_{3,1})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,2})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,3})})} \\ -\frac{e^{-j2\pi f\tau_0\alpha_{3,1}}+e^{-j2\pi f\tau_0\alpha_{3,2}}+e^{-j2\pi f\tau_0\alpha_{3,3}}}{(1-e^{j2\pi f\tau_0(1-\alpha_{3,1})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,2})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,3})})} \\ \frac{e^{-j2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,2})}+e^{-j2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,3})}+e^{-j2\pi f\tau_0(\alpha_{3,2}+\alpha_{3,3})}}{(1-e^{j2\pi f\tau_0(1-\alpha_{3,1})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,2})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,3})})} \\ -\frac{e^{-j2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,2}+\alpha_{3,3})}}{(1-e^{j2\pi f\tau_0(1-\alpha_{3,1})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,2})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,3})})} \end{bmatrix}$$

$$h_3(f) = \frac{1}{(1-e^{j2\pi f\tau_0(1-\alpha_{3,1})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,2})})(1-e^{j2\pi f\tau_0(1-\alpha_{3,3})})} \begin{bmatrix} 1 \\ -e^{-j2\pi f\tau_0\alpha_{3,1}} - e^{-j2\pi f\tau_0\alpha_{3,2}} - e^{-j2\pi f\tau_0\alpha_{3,3}} \\ e^{-j2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,2})} + e^{-j2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,3})} + e^{-j2\pi f\tau_0(\alpha_{3,2}+\alpha_{3,3})} \\ -e^{-j2\pi f\tau_0(\alpha_{3,1}+\alpha_{3,2}+\alpha_{3,3})} \end{bmatrix}$$

■

9.16

Show that in the case of a third-order DSA with a zero of multiplicity 3 in the beampattern, the beamformer is given by

$$\mathbf{h}_{3,0}(f) = \frac{1}{\left[1-e^{j2\pi f\tau_0(1-\alpha_{3,1})}\right]^3} \begin{bmatrix} 1 \\ -3e^{-j2\pi f\tau_0\alpha_{3,1}} \\ 3e^{-j4\pi f\tau_0\alpha_{2,1}} \\ -e^{-j6\pi f\tau_0\alpha_{2,1}} \end{bmatrix},$$

Solution:
we want to solve the following linear problem:

$$\begin{bmatrix} d^H(f,1) \\ d^H(f,\alpha_{3,1}) \\ \sum d^H(f,\alpha_{3,2})^H \\ \sum{}^2 d^H(f,\alpha_{3,3})^H \end{bmatrix} h(f)$$

so:

$$\begin{bmatrix} d^H(f,1) \\ d^H(f,\alpha_{3,1}) \\ \sum d^H(f,\alpha_{3,2})^H \\ \sum{}^2 d^H(f,\alpha_{3,3})^H \end{bmatrix} h(f) = \begin{bmatrix} 1 & v_1 & {v_1}^2 & {v_1}^3 \\ 1 & v_2 & {v_2}^2 & {v_2}^3 \\ \multicolumn{4}{c}{\sum d^H(f,\alpha_{3,2})^H} \\ \multicolumn{4}{c}{\sum{}^2 d^H(f,\alpha_{3,3})^H} \end{bmatrix} h_{3,0}(f) = \begin{bmatrix} [1 \quad v_1 \quad {v_1}^2 \quad {v_1}^3]h_{3,0}(f) \\ [1 \quad v_2 \quad {v_2}^2 \quad {v_2}^3]h_{3,0}(f) \\ \sum d^H(f,\alpha_{3,2})^H h_{3,0}(f) \\ \sum{}^2 d^H(f,\alpha_{3,3})^H h_{3,0}(f) \end{bmatrix} = (a)$$

from 9.86 we know:

$$j2\pi f\tau_0 \sum d^H(f,\alpha_{3,2})^H h_{3,0}(f) = 0 \to \sum d^H(f,\alpha_{3,2})^H h_{3,0}(f) = 0$$

and from 9.87 we know:

$$(j2\pi f\tau_0)^2 \sum{}^2 d^H(f,\alpha_{3,3})^H h_{3,0}(f) = 0 \to \sum{}^2 d^H(f,\alpha_{3,3})^H h_{3,0}(f) = 0$$

substituting:

$$(a) = \frac{1}{\left[1-e^{j2\pi f\tau_0(1-\alpha_{3,1})}\right]^3} \begin{bmatrix} [1 \quad e^{j2\pi f\tau_0} \quad e^{j4\pi f\tau_0} \quad e^{j6\pi f\tau_0}] \begin{bmatrix} 1 \\ -3e^{-j2\pi f\tau_0\alpha_{3,1}} \\ 3e^{-j4\pi f\tau_0\alpha_{3,1}} \\ -e^{-j6\pi f\tau_0\alpha_{3,1}} \end{bmatrix} \\ [1 \quad e^{j2\pi f\tau_0\alpha_{3,1}} \quad e^{j4\pi f\tau_0\alpha_{3,1}} \quad e^{j6\pi f\tau_0\alpha_{3,1}}] \begin{bmatrix} 1 \\ -3e^{-j2\pi f\tau_0\alpha_{3,1}} \\ 3e^{-j4\pi f\tau_0\alpha_{3,1}} \\ -e^{-j6\pi f\tau_0\alpha_{3,1}} \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} =$$

$$= \frac{1}{\left[1-e^{\jmath 2\pi f\tau_0(1-\alpha_{3,1})}\right]^3}\begin{bmatrix} 1-3e^{-\jmath 2\pi f\tau_0\alpha_{3,1}}e^{\jmath 2\pi f\tau_0}+3e^{-\jmath 4\pi f\tau_0\alpha_{3,1}}e^{\jmath 4\pi f\tau_0}-e^{-\jmath 6\pi f\tau_0\alpha_{3,1}}e^{\jmath 6\pi f\tau_0} \\ 1-3e^{-\jmath 2\pi f\tau_0\alpha_{3,1}}e^{\jmath 2\pi f\tau_0\alpha_{3,1}}+3e^{-\jmath 4\pi f\tau_0\alpha_{3,1}}e^{\jmath 4\pi f\tau_0\alpha_{3,1}}-e^{-\jmath 6\pi f\tau_0\alpha_{3,1}}e^{\jmath 6\pi f\tau_0\alpha_{3,1}} \\ 0 \\ 0 \end{bmatrix} =$$

$$= \frac{1}{\left[1-e^{\jmath 2\pi f\tau_0(1-\alpha_{3,1})}\right]^3}\begin{bmatrix} 1-3e^{\jmath 2\pi f\tau_0(1-\alpha_{3,1})}+3e^{\jmath 4\pi f\tau_0(1-\alpha_{3,1})}-e^{\jmath 6\pi f\tau_0(1-\alpha_{3,1})} \\ 1-3+3-1 \\ 0 \\ 0 \end{bmatrix} =$$

$$= \frac{1}{\left[1-e^{\jmath 2\pi f\tau_0(1-\alpha_{3,1})}\right]^3}\begin{bmatrix} \left[1-e^{\jmath 2\pi f\tau_0(1-\alpha_{3,1})}\right]^3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

■

the beampattern can be approximated as

$$\mathcal{B}\left[\mathbf{h}_{3,0}(f),\cos\theta\right] \approx \frac{1}{\left(1-\alpha_{3,1}\right)^3}\left(\cos\theta-\alpha_{3,1}\right)^3,$$

Solution:
the beampattern has the form:

$$B[h_{3,0}(f),\cos\theta] = \frac{\left[1-e^{j2\pi f\tau_0(\cos\theta-\alpha_{3,1})}\right]^3}{\left[1-e^{j2\pi f\tau_0(1-\alpha_{3,1})}\right]^3}$$

using the following approximation:

$$e^x \approx 1+x$$

we get:

$$B[h_{3,0}(f),\cos\theta] \approx \frac{\left[1-1-j2\pi f\tau_0(\cos\theta-\alpha_{3,1})\right]^3}{\left[1-1-j2\pi f\tau_0(1-\alpha_{3,1})\right]^3} = \frac{\left[j2\pi f\tau_0(\cos\theta-\alpha_{3,1})\right]^3}{\left[j2\pi f\tau_0(1-\alpha_{3,1})\right]^3} = \frac{\left[\cos\theta-\alpha_{3,1}\right]^3}{\left[1-\alpha_{3,1}\right]^3}$$

■

and the WNG can be approximated as

$$\mathcal{W}\left[\mathbf{h}_{3,0}(f)\right] \approx \frac{1}{20}\left[2\pi f\tau_0\left(1-\alpha_{3,1}\right)\right]^6.$$

Solution:
We find that the WNG is:

$$W[h(f)] = \frac{1}{20}\left|1-e^{j2\pi f\tau_0(1-\alpha_{3,1})}\right|^6 = \frac{2}{5}[1-\cos[2\pi f\tau_0(1-\alpha_{3,1})]]^3$$

with the following approximation:

$$\cos x \approx 1-\frac{x^2}{2}$$

The WNG can be approximated as

$$W[h(f)] \approx \frac{2}{5}\left[1-1+\frac{[2\pi f\tau_0(1-\alpha_{3,1})]^2}{2}\right]^3 = \frac{2}{5\cdot 8}[2\pi f\tau_0(1-\alpha_{3,1})]^6 = \frac{1}{20}[2\pi f\tau_0(1-\alpha_{3,1})]^6$$

■

9.18

Show that the beampattern, the WNG, and the DF of the minimum-norm beamformer are given by

$$\mathcal{B}\left[\mathbf{h}_{\mathrm{MN}}\left(f,\boldsymbol{\alpha},\boldsymbol{\beta}\right),\cos\theta\right]=\mathbf{d}^{H}\left(f,\cos\theta\right)\mathbf{D}^{H}\left(f,\boldsymbol{\alpha}\right)\times$$
$$\left[\mathbf{D}\left(f,\boldsymbol{\alpha}\right)\mathbf{D}^{H}\left(f,\boldsymbol{\alpha}\right)\right]^{-1}\boldsymbol{\beta},$$
$$\mathcal{W}\left[\mathbf{h}_{\mathrm{MN}}\left(f,\boldsymbol{\alpha},\boldsymbol{\beta}\right)\right]=\frac{1}{\boldsymbol{\beta}^{T}\left[\mathbf{D}\left(f,\boldsymbol{\alpha}\right)\mathbf{D}^{H}\left(f,\boldsymbol{\alpha}\right)\right]^{-1}\boldsymbol{\beta}},$$
$$\mathcal{D}\left[\mathbf{h}_{\mathrm{MN}}\left(f,\boldsymbol{\alpha},\boldsymbol{\beta}\right)\right]=\frac{1}{\mathbf{h}_{\mathrm{MN}}^{H}\left(f,\boldsymbol{\alpha},\boldsymbol{\beta}\right)\boldsymbol{\Gamma}_{0,\pi}(f)\mathbf{h}_{\mathrm{MN}}\left(f,\boldsymbol{\alpha},\boldsymbol{\beta}\right)}.$$

Solution:
First we know:

$$(1)h_{MN}(f,\alpha,\beta)=D^{H}(f,\alpha)[D(f,\alpha)D^{H}(f,\alpha)]^{-1}\beta$$
$$(2)h_{MN}(f,\alpha,\beta)^{H}d(f,1)=1$$

It is easy to see that the beampattern is:

$$B[h_{MN}(f,\alpha,\beta),\cos\theta]=d^{H}(f,\cos\theta)h_{MN}(f,\alpha,\beta)=$$
$$=d^{H}(f,\cos\theta)D^{H}(f,\alpha)[D(f,\alpha)D^{H}(f,\alpha)]^{-1}\beta$$

■

The WNG defined as:

$$W[h(f)]=\frac{\left|h^{H}(f)d(f,1)\right|^{2}}{h^{H}(f)h(f)}$$

substitute (1) and (2) to the definition of WNG:

$$W[h_{MN}(f)]=\frac{\left|{h_{MN}}^{H}(f,\alpha,\beta)d(f,1)\right|^{2}}{{h_{MN}}^{H}(f,\alpha,\beta)h_{MN}(f,\alpha,\beta)}=$$
$$=\frac{1^{2}}{\beta^{H}(f,\alpha)[D(f,\alpha)D^{H}(f,\alpha)]^{-1}[D(f,\alpha)D^{H}(f,\alpha)][D(f,\alpha)D^{H}(f,\alpha)]^{-1}\beta}=$$
$$=\frac{1}{\beta^{H}(f,\alpha)[D(f,\alpha)D^{H}(f,\alpha)]^{-1}\beta}$$

■

The DF of the minimum-norm beamformer defined as:

$$D[h(f)]=\frac{\left|h^{H}(f)d(f,1)\right|^{2}}{h^{H}(f)\Gamma_{0,2\pi}h(f)}$$

substitute (1) and (2) to the definition of the DF;

$$D[h_{Mn}(f,\alpha,\beta)]=\frac{\left|h_{Mn}(f,\alpha,\beta)^{H}d(f,1)\right|^{2}}{h_{Mn}(f,\alpha,\beta)^{H}\Gamma_{0,2\pi}h_{Mn}(f,\alpha,\beta)}=\frac{1}{h_{Mn}(f,\alpha,\beta)^{H}\Gamma_{0,2\pi}h_{Mn}(f,\alpha,\beta)}$$