Calculus in a Nutshell

"The only way to LEARN mathematics
is to DO mathematics"–Paul Halmos

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Name: ___________________________

Let's do some Calculus.
I bet you can't find the minion. ⌣̈

Calculus in a Nutshell

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1 Limits

1.1 What is a limit?

Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval, (h,k), where $h < a < k$ except possibly $f(a)$.) Then we write:

$$\lim_{x \to a} f(x) = L$$

and say "The limit of $f(x)$ as x approaches a, equals L."

Basically a limit is the number that the function "wants" to be, but when you plug in an x-value, we get an undefined answer. $\left(\frac{x}{0} \text{ where } x \neq 0, \text{ or } \frac{0}{0}\right)$
There are 3 ways to find limits:

1. Tables
2. Graphs
3. Algebraically (our favorite $\smile$)

Algebraic limits are the most difficult to solve, and it can involve any of the following methods.

1. Factoring
2. Multiplying by the conjugate
3. Common denominators
4. Multiply by the reciprocal

Let's try one!

1.2 Example

$$\lim_{t \to 1} \frac{t^4 - 1}{t^3 - 1}$$

First off, we can notice that we can't just plug in 1 because then we would get $\frac{0}{0}$, and we can't divide by 0. To fix this, we must algebraically rearrange the fraction so we can plug in 1. Let's start by factoring.

$$\lim_{t \to 1} \frac{(x^2+1)(x+1)(x-1)}{(x-1)(x^2+x+1)}$$

This is great because now we can cancel $(x-1)$ from the numerator and denominator.

$$\lim_{t \to 1} \frac{(x^2+1)(x+1)}{(x^2+x+1)}$$

Now if we plug in 1 we get $\frac{4}{3}$. This means we can say the limit is $\frac{4}{3}$!

1.3 Conceptual ideas about limits

1. When we plug in a number to a function and get $\frac{x}{0}$ where $x \neq 0$, we have a vertical **asymptote**.

2. If we plug in a number to a function and get $\frac{0}{0}$ we have a **hole** in our graph.

1.4 More Limit notation

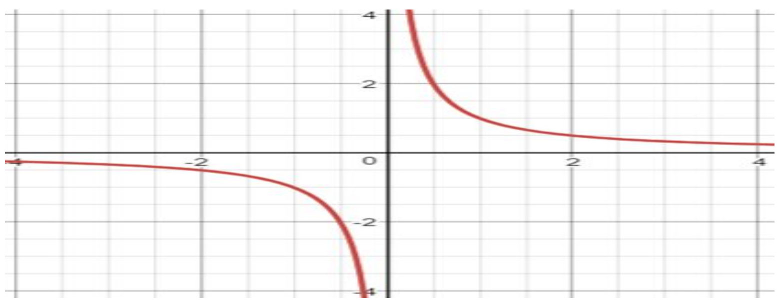
$$\lim_{x \to a^+} f(x)$$

This is read "The limit of $f(x)$ as x approaches a from the right side, equals L."

$$\lim_{x \to a^-} f(x)$$

This is read "The limit of $f(x)$ as x approaches a from the left side, equals L."

You might be wondering, "Why do we have left and right limits?" If you look at the graph of $f(x) = \frac{1}{x}$, you might understand.



As you can see, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$, and $\lim_{x \to 0^+} \frac{1}{x} = \infty$, and thus a distinction needs to be made

2 Continuity

We can say a function is continuous at a point a if $(a, f(a))$ exists, and

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

In other words, if a function connects from both sides **and** you can actually plug in a to the function.

A function is continuous from the right at a if $(a, f(a))$ exists and

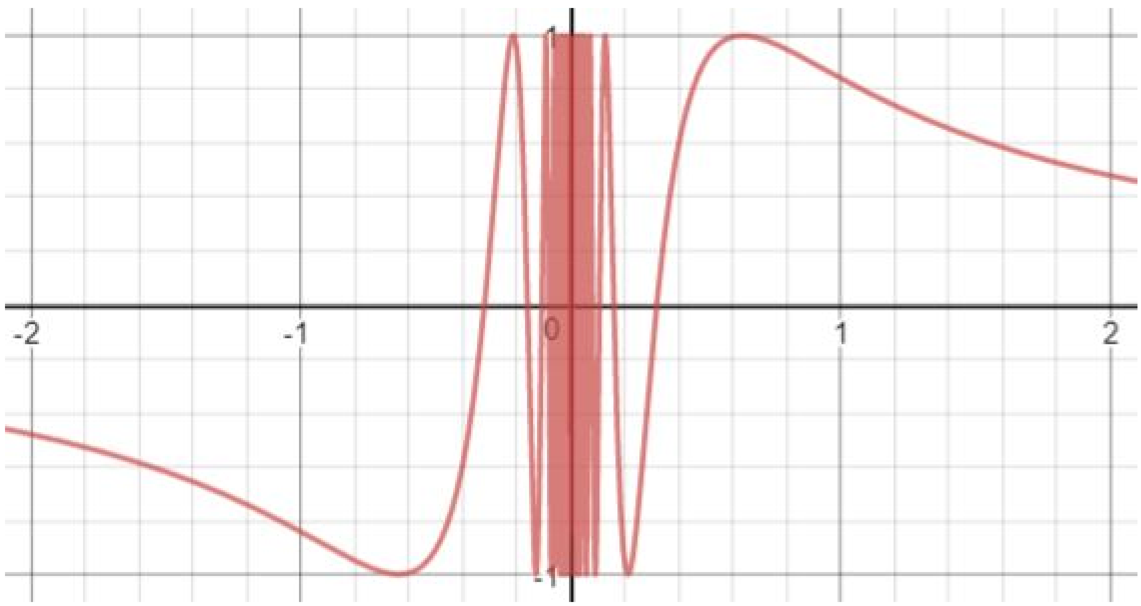
$$\lim_{x \to a^+} f(x) = f(a)$$

A function is continuous from the left at a if $(a, f(a))$ exists and

$$\lim_{x \to a^-} f(x) = f(a)$$

We have 4 types of discontinuities.

1. Removable-In this discontinuity, $\lim_{x\to a} f(x) = L$. $f(a)$ might exist, but $f(a) \neq L$.
2. Infinite-In this discontinuity, $\lim_{x\to a} f(x) = \pm\infty$, and $(a, f(x))$ does not exist.
3. Jump-In this discontinuity, $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. $f(a)$ might exist.
4. Oscillating-In this discontinuity, the function oscillates infinitely many times in a finite distance, and thus is impossible to find the limit. An example is $f(x) = \sin\left(\frac{1}{x}\right)$



3 Limits involving infinity

When we talk about limits involving infinity, we are talking about what a function does as x goes to positive or negative infinity. We have two options, either the function just keeps getting bigger (smaller) and goes off to infinity (negative infinity), or the function gets closer to one specific number. We call this number a horizontal asymptote. To find a horizontal asymptote we need to find out what $f(x)$ is doing as it goes out to $\pm\infty$.

In other words we need to take the limit as our function goes to positive or negative infinity.

$$\lim_{x\to\infty} f(x)$$

To be more formal we can let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of the function can be made arbitrarily close to L by requiring x to be sufficiently large. **You can flip this definition to work for $-\infty$ too.**

Now that we can find our limit (L) there is one thing that we need to realize. Our horizontal asymptote is a **horizontal line** not just a value. Therefore, the asymptote is not L, but $x = L$.

We know we need to, but how do we take a limit to infinity?
I am glad you asked!
All we do is divide every term by the biggest power in the denominator, and simplify from there.

**Note that in the following problem, $\lim\limits_{x\to\infty} \frac{1}{x^n}$ will come up. If we think about this, the denominator will get very large, and thus making $\frac{1}{x^n}$ very small. Thus, if $n > 0$, $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ (this is true no matter what constant is on top).

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

x^2 is the largest power in the denominator

so we will divide every term by x^2

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}}$$

We simplify each term with algebra.

$$= \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

Remember the note.

$$= \frac{3-0-0}{5+0+0}$$

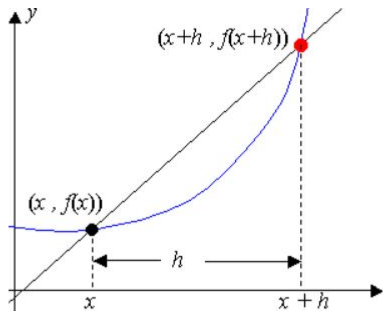
$$= \frac{3}{5}$$

Therefore our horizontal asymptote is $x = \frac{3}{5}$

4 Tangent lines and derivatives.

A tangent line is a line that touches our function f at one and only one point. It touches our graph and skips off.

As you already know, the equation for the slope between 2 points (secant line) is $m = \frac{y_1 - y_0}{x_1 - x_0}$. Now let's say $P_0 = (a, f(a))$, and $P_1 = (a+h, f(a+h))$. All h is saying is that our second point is a horizontal distance away from our origional point (Look at the picture).



If we write our slope equation with these two points, we can say

$$m = \frac{f(a+h) - f(a)}{(a+h) - a}$$

We can simplify the denominator to be

$$m = \frac{f(a+h) - f(a)}{h}$$

As we know, slope needs rise **and** run, but in a tangent line, there is no run. As we can see, our h is the run. Since a tangent line has no run, what if we take the limit as h goes to 0?

$$m_{\text{ tangent line}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

In calculus, we like to call the slope of the tangent line the **DERIVATIVE**!

Now let's try finding the slope of the tangent line with some numbers.

$$f(x) = x^2 - 3x + 6, \text{ at } (a, f(a)) = (2, 4)$$

This would mean $(a+h) = (2+h)$ and $f(a+h) = f(2+h)$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 3(2+h) + 6 - 4}{h}$$

Expand everything.

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 6 - 3h + 6 - 4}{h}$$

Cancel/combine like terms.

$$= \lim_{h \to 0} \frac{4h + h^2 - 3h}{h}$$

Factor out an h

from every term in the numerator.

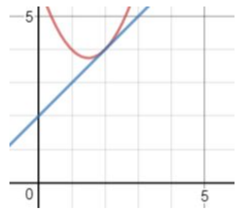
$$= \lim_{h \to 0} \frac{h(4 + h - 3)}{h}$$

$$= \lim_{h \to 0} 4 + h - 3$$

$$= 1$$

If we use point slope form, with a slope of 1 and a point of (2,4), we can say the tangent line is

$$y-4=1(x-2).$$



Questions:

1. Can we find a function, call it $f'(x)$, so that we can plug in any value of x and get the slope of the tangent line at that point?
2. What happens if we let a be x? (Remember a stands for a specific number and x is generalized)

Let's try the same function as before. $f(x)=x^2-3x+6$

$$\begin{aligned}&\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ =&\lim_{h\to 0}\frac{(x+h)^2-3(x+h)+6-(x^2-3x+6)}{h}\\ =&\lim_{h\to 0}\frac{x^2+2xh+h^2-3x-3h+6-x^2+3x-6}{h}\\ =&\lim_{h\to 0}\frac{2xh+h^2-3h}{h}\\ =&\lim_{h\to 0}\frac{h(2x+h-3)}{h}\\ =&2x-3\end{aligned}$$

Let's name this function $f'(x)$ (read as "'f' prime of 'x'"). This means $f'(x)=2x-3$. If we plug in $x=2$, we should get 1 (because we originally let $a=1$ the first time).

$$f'(2)=2(2)-3=1$$

It worked!!! You can check this for any value for x, and it will always work out! $f'(x)$ is the derivative of $f(x)$ because when we plug in a value for x in $f'(x)$, we will get the slope of $f(x)$, and that is the derivative!

Basically, the big take away from this is the definition of the derivative is

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

and the general form of the derivative is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

5 Higher order derivatives

A higher order derivative is the derivative of the derivative. So basically, we like taking derivatives so much that we do it again and again!!!

Let's start with a function $f(x) = \sqrt{x^3}$. If we want to find the second derivative of $f(x)$, or $f''(x)$, then we need to start with taking the first derivative of the function, or $f'(x)$. To take the derivative, we need to again take the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$
\begin{aligned}
&\lim_{h\to 0} \frac{\sqrt{(x+h)^3}-\sqrt{x^3}}{h} \\
&\text{Multiply by conjugate.} \\
=&\lim_{h\to 0} \frac{(x+h)^3-x^3}{h(\sqrt{(x+h)^3}+\sqrt{x^3})} \\
&\text{Expand.} \\
=&\lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h(\sqrt{(x+h)^3}+\sqrt{x^3})} \\
&\text{Combine like terms.} \\
=&\lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h(\sqrt{(x+h)^3}+\sqrt{x^3})} \\
&\text{Factor and divide by } h \\
=&\lim_{h\to 0} \frac{3x^2+3x^2+h^2}{\sqrt{(x+h)^3}+\sqrt{x^3}} \\
=&\frac{3x^{\frac{1}{2}}}{2} = f'(x)
\end{aligned}
$$

Now to find the second derivative we get to do that again except this time we use $\frac{3x^{\frac{1}{2}}}{2} = f'(x)$!!!

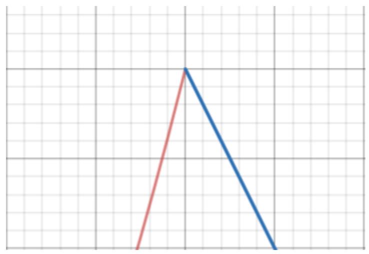
$$
\begin{aligned}
&\lim_{h\to 0} \frac{\frac{3}{2}(x+h)^{\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}}{h} \\
=&\lim_{h\to 0} \frac{\frac{3}{2}(\sqrt{x+h}-\sqrt{x})}{h} \\
=&\lim_{h\to 0} \frac{\frac{3}{2}(x+h-x)}{h(\sqrt{x+h}+\sqrt{x})} \\
=&\lim_{h\to 0} \frac{\frac{3}{2}h}{h(\sqrt{x+h}+\sqrt{x})} \\
=&\frac{3}{4\sqrt{x}} = f''(x)
\end{aligned}
$$

We can continue to take the next derivative as many times as we want. One quick note on notation though, $f'(x) = f^1(x)$, and $f''(x) = f^2(x)$, and so on.

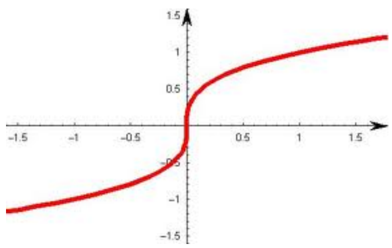
6 What can we differentiate?

We can't differentiate everything. This section talks about times when we can't find a derivative.

1. Whenever there is a corner in the graph we can't find a derivative.



2. If there is a vertical tangent line there is no derivative because the slope is undefined.



3. There is also no derivative at any discontinuity (our function must be continuous).

7 Derivative notation

Now that we are working with derivatives, we need to know some notation.

Leibniz notation: $\frac{d}{dx}(f(x))$ is a way to say the derivative with respect to 'x'." In this notation, the x can be any variable. Let's say we have $\frac{d}{dy}y^2$, we would just treat y as our variable. If we have something like this: $\frac{d}{dx}y^2x^2$, we would just treat y as a constant. This will make more sense as we go.

8 Basic derivative rules

As you could imagine, using the definition of the derivative all the time would be very annoying, and complicated functions could be very easy to mess up. Don't worry, we have shortcuts!!!

1. **Constant Rule:** The derivative of a constant is 0.

$$\frac{d}{dx}(c) = 0$$

2. **Power Rule:** $\frac{d}{dx}x^2 = 2x$, in general,

$$\frac{d}{dx}(x^n) = nx^n - 1$$

3. **Constant Multiple Rule:** We can factor out a constant before we take our derivative. $\dfrac{d}{dx}2x^2 = 2\dfrac{d}{dx}x^2 = 2(2x) = 4x$.
$$\frac{d}{dx}\big(cf(x)\big) = c\frac{d}{dx}f(x)$$
4. **Sum and Difference Rules:** We can split up derivatives at plus and minus signs, and take the derivatives of each term.
$$\frac{d}{dx}\big(f(x)+g(x)\big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$
$$\frac{d}{dx}\big(f(x)-g(x)\big) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$
5. **Natural exponential function:** $\dfrac{d}{dx}e^x = e^x$

Let's find a derivative!

$$\frac{d}{dx}\left(\frac{x^2+x-2}{x+2}\right)$$

We have a problem because we have a fraction, and we can't do anything with that. Let's try to rearrange it so it ends up being in a line.

$$\begin{aligned}&\frac{d}{dx}\left(\frac{x^2+x-2}{x+2}\right)\\ =&\frac{d}{dx}\left(\frac{(x-1)(x+2)}{x+2}\right)\\ =&\frac{d}{dx}x^1-1\\ =&1\end{aligned}$$

At the end, we used the difference, power, and constant rules.

9 Specialized derivative rules

9.1 Product Rule

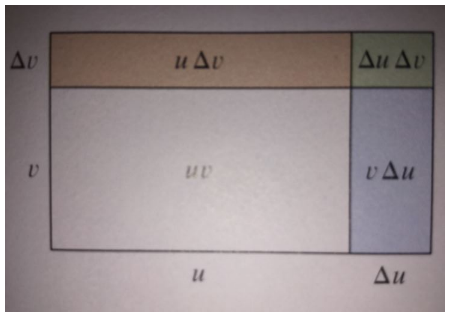
Now that you are familiar with what a derivative is, we can start looking at some special situations that will come up. One of these situations is when we have a function times a function, or something along the lines of $\frac{d}{dx}(f(x)g(x))$. These functions need to be something we can't combine. If we had $x^2(x^3)$, we could just say that is x^5, and then just use power rule. However, if we have something like xe^x, we can't combine those. If you use the definition of the derivative (the limit thing), you would find the following:

$$\frac{d}{dx}xe^x = xe^x + e^x$$

This is very different than this:

$$\left(\frac{d}{dx}x\frac{d}{dx}e^x\right) = 1e^x = e^x$$

Clearly, somewhere along the line, something happened. This picture shows in terms of geometry what is happening visually.



The first method we tried (the definition of the derivative) found the colored area, and the second only found the green corner.

The formula for product rule is $\frac{d}{dx}(fg) = fg' + gf'$. Here's an example:

$$\frac{d}{dx}(xe^x)$$

First off, we have to recognize which function is f and which is g. With the product rule, it doesn't actually matter.

$f = x$	$g = e^x$
$f' = 1$	$g' = e^x$

Now that we have our 4 functions, we just follow the formula.

$$\frac{d}{dx}(xe^x) = x(e^x) + (e^x)1 = xe^x + e^x$$

This is what we got with the definition of the derivative!

9.2 Quotient Rule

A second special situation we have is a function divided by a function, or something along the lines of $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$. These functions need to be something we can't combine easily. If we had $\frac{x^2}{x^3}$, we could just say that is $\frac{1}{x} = x^{-1}$, and then just use power rule. However, if we have something like $\frac{x^2+x-2}{x^3+6}$, we can't combine those easily. If you use the definition of the derivative, you would find the following:

$$\frac{d}{dx}\left(\frac{x^2+x-2}{x^3+6}\right) = \frac{(x^3+6)(2x+1)-(x^2+x-2)(3x^2)}{(x^3+6)^2}$$

This is very different than this:

$$\left(\frac{\frac{d}{dx}(x^2+x-2)}{\frac{d}{dx}(x^3+6)}\right) = \frac{2x+1}{3x^2}$$

Clearly, somewhere along the line, something happened.
The formula we have for quotient rule is $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$. Here is an example:

$$\frac{d}{dx}\left(\frac{x^2+x-2}{x^3+6}\right)$$

First off, just like product rule we have to recognize which function is f and which is g. In quotient rule, the numerator is ALWAYS f and the denominator is ALWAYS g. So $\dfrac{f}{g}$.

$f = x^2+x-2$	$g = x^3+6$
$f' = 2x+1$	$g' = 3x^2$

Now that we have our 4 functions, we just follow the formula.

$$\frac{d}{dx}\left(\frac{x^2+x-2}{x^3+6}\right) = \frac{(x^3+6)(2x+1)-(x^2+x-2)(3x^2)}{(x^3+6)^2}$$

9.2.1 Derivatives of trigonometric functions USING QUOTIENT RULE

What are trig functions?
TRIG IDENTITIES:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

Derivatives of trig functions can be a lot to remember, but if you can remember these identities, you can always use quotient rule to find them. Two derivatives you have to remember though are $\sin x$ and $\cos x$.

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$

If you know these two, we can extrapolate our list to become this:
(let "$\rightarrow$" be a derivative)

$$\overleftarrow{\sin x \rightarrow \cos x \rightarrow -\sin x \rightarrow -\cos x}$$

Basically, this picture shows how the derivative of $\sin x$ and $\cos x$ will loop forever.

Combining this chart, quotient rule, and the trig identities shown above, we get the following table.

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$

9.3 Chain Rule

A third special situation we have is a composite function, or something along the lines of $\frac{d}{dx}(f(g(x)))$. These kinds of function can't be combined. If we have something like $\sqrt{x^2+1}$, and we use the definition of the derivative, you would find the following:

$$\frac{d}{dx}\left(\sqrt{x^2+1}\right) = \frac{x}{\sqrt{x^2+1}}$$

This is very different than this:

$$\frac{d}{dx}\left(\sqrt{\frac{d}{dx}x^2+1}\right) = \frac{1}{\sqrt{2x}}$$

The formula we have for chain rule is $f'(g(x))g'(x)$. Here's an example:

$$\frac{d}{dx}\left(\sqrt{x^2+1}\right)$$

First off, just like product and quotient rule, we have to recognize which function is f and which is g. With composite functions, the "outside" function is f, and the "inside" function is g. Another way to decide which is f and which is g is that g is the function you would have to evaluate first in order to solve for a value, and f is the second function. So with $\sqrt{x^2+1}$, x^2+1 is inside the square root function ($\sqrt{\ }$). You would also have to evaluate x^2+1 first before you could find the square root of the value. Therefore, $f=\sqrt{x}$ and $g=x^2+1$.

$f=\sqrt{x}$	$g=x^2+1$
$f'=\dfrac{1}{2\sqrt{x}}$	$g'=2x$

Following our formula we get

$$\frac{d}{dx}\left(\sqrt{x^2+1}\right)=\frac{1}{2\sqrt{x^2+1}}2x=\frac{2x}{2\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}}$$

10 Exponential Functions

Exponential functions are written in the form $f(x)=b^x$. To take the derivative of an exponential function like 2^x for example, we must first realize that we can rewrite 2^x as $e^{(\ln 2)x}$. We can so this because $e^{\ln x}=x$. Also because of exponent rules, we can then write $(\ln 2)^x$ as $(\ln 2)x$. Let's take it from this point: $e^{(\ln 2)x}$.

Using chain rule:

$$\begin{aligned}
&e^{(\ln 2)x}\\
f=e^x \quad &g=(\ln 2)x\\
f'=e^x \quad &g'=\ln 2\\
&f'(g(x))g'(x)\\
&e^{(\ln 2)x}\ln 2\\
&=2^x\ln 2
\end{aligned}$$

In general, $\dfrac{d}{dx}(b^x)=b^x\ln b$.

11 Implicit Differentiation

We have been taking the derivatives of functions with respect to one variable (most of the time x) by using Leibniz notation. For example, $\dfrac{d}{dx}(x^2)$ is read as "the derivative of x squared with respect to x." What happens if we have to take the derivative of a function that has 2 variables? What if we have to take the derivative of $x^2+y^2=1$ (this is the equation for a circle centered at the origin with a radius of 1 [the unit circle])?

This is where Leibniz notation plays a bigger role. Just remember, $\dfrac{d}{dx}$ is the derivative **WITH RESPECT TO** x. Let's just work through this derivative as I explain.

$$
\begin{aligned}
&\frac{d}{dx}(x^2+y^2=1)\\
=&\frac{d}{dx}x^2+\frac{d}{dx}y^2=\frac{d}{dx}1\\
=&2x+\frac{d}{dx}y^2=0 \text{ (We aren't done yet)}
\end{aligned}
$$

Let's talk a little more about Leibniz notation. We have seen $\frac{d}{dx}$ a lot so far, but we are going to add something to our notation. If we have $\frac{dy}{dx}$, we say "The derivative of y with respect to x." This is saying that y is a function of x. In other words, we are implying $y(x)$ is a function even though we don't know what $y(x)$ equals.

Now let's worry about the "weird" part. What is $\frac{d}{dx}y^2$? Whenever we take a derivative with respect to x ($\frac{d}{dx}$) of a variable other than x (y in this case) we need to multiply by the derivative of that variable (y in this case) with respect to x ($\frac{dy}{dx}$ in this case). Think of y as a function $y(x)$.

Observe:

$$
\begin{aligned}
&\frac{d}{dx}(y^2)\\
=&\frac{dy}{dx}2y
\end{aligned}
$$

Basically, we take the derivative of y like normal ($y^2 \to 2y$), and then multiply by $\frac{dy}{dx}$ which is read "the derivative of y with respect to x." In a sense we are treating y as a function inside the larger function (that sounds a lot like the chain rule). Now we have to finish up our problem.

We now have $2x+2y\frac{dy}{dx}=0$, and we need to solve for $\frac{dy}{dx}$ because we want to know the derivative of y.

$$
\begin{aligned}
2x+2y\frac{dy}{dx} =&0\\
2y\frac{dy}{dx} =&-2x\\
\frac{dy}{dx} =&\frac{-2x}{2y}\\
\frac{dy}{dx} =&\frac{-x}{y}
\end{aligned}
$$

Great! We solved the problem! What does the answer tell us though? If we go back to the original equation of our circle, $x^2+y^2=1$, we know we had a circle with a radius of 1. By taking the derivative, we were finding the slope of the tangent line at any point of the circle, but since we had 2 variables we can't explicitly define the slope with one variable, instead, we had to implicitly define it with 2.

Remember when I said think of y as a function in terms of x? We didn't know the function, and we still don't, but what if just for kicks and giggles, we said $y=3x^2+x$. In our equation we had a y^2. Well, if we substitute $3x^2+x$ in for y we have $y(x)=(3x^2+x)^2$. This looks awful lot like chain rule, in which case we do the following:

$$
\begin{aligned}
&\frac{d}{dx}(3x^2+x)^2 \\
f = x^2 &\quad y = 3x^2+x \\
f' = 2x &\quad y' = 6x+1 \\
&\frac{d}{dx} = 2(3x^2+x)(6x+1)
\end{aligned}
$$

Now if we look at our answer, and put it back in terms of y (remember $y = 3x^2+x$, and $y' = 6x+1$) we get

$$2y(y').$$

Which is the same as $2y\frac{dy}{dx}$.

We can say that $y' = \frac{dy}{dx}$ because y' is the derivative of y with respect to x because y was in terms of x. When we do implicit differentiation, we are basically doing a speed version of the chain rule because y (or z, or r, or any other variable) is itself a function in terms of x.

Let's try one more example.

$$\frac{d}{dx}(y^4+xy = x^3-x+2)$$

$$
\begin{aligned}
\frac{d}{dx}y^4+\frac{d}{dx}(xy) &= \frac{d}{dx}x^3+\frac{d}{dx}2 \\
\frac{d}{dx}y^4+\frac{d}{dx}(xy) &= 3x^2
\end{aligned}
$$

Now that we took the basic derivatives, let's look at what we have left.

$$\frac{d}{dx}y^4$$

First we have to remember that y represents an equation in terms of x. Just like x can be 3 in one equation and 18 in another, y can represent whatever it wants. For kicks and giggles, what if we make up something that y represents. What happens if $y = x^2+x$.

$$\frac{d}{dx}y^4 = \frac{d}{dx}(x^2+x)^4$$

$$\Uparrow$$

This looks like chain rule

$$
\begin{aligned}
f'(g(x))g'(x)\\
f = x^4 \;\; g = x^2 + x\\
f' = 4x^3 \;\; g' = 2x + 1
\end{aligned}
$$

$$
\begin{aligned}
4(x^2+x)^3(2x+1) =& 4y^3y'\\
=& 4y^3\left(\frac{dy}{dx}\right)
\end{aligned}
$$

As we can see if we take the derivative of y like normal, and then multiply by y' which is equal to $\frac{dy}{dx}$ we are just doing the chain rule quickly. REMEMBER we DON'T actually know what y is. That's why we need to keep it as y.

We have one last derivative to take

$$\frac{d}{dx}(xy)$$

If we look at this, we can see that if y is a function in terms of x, we have a function times a function (product rule).

$$(fg') + (gf')$$

$$
\begin{aligned}
f = x \;\; g = y\\
f' = 1 \;\; g' = 1\frac{dy}{dx} = \frac{dy}{dx}\\
\frac{d}{dx}(xy) =& x\frac{dy}{dx} + y
\end{aligned}
$$

Now we have all our parts of our equation, and we can solve for $\frac{dy}{dx}$

$$
\begin{aligned}
4y^3\left(\frac{dy}{dx}\right) + x\left(\frac{dy}{dx}\right) + y =& 3x^2\\
4y^3\left(\frac{dy}{dx}\right) + x\left(\frac{dy}{dx}\right) =& 3x^2 - y\\
\frac{dy}{dx}(4y^3 + x) =& 3x^2 - y\\
\frac{dy}{dx} =& \frac{3x^2 - y}{4y^3 + x}
\end{aligned}
$$

12 Natural log function

We can find the derivative of $\ln x = \frac{1}{x}$ using implicit differentiation.

$$\begin{aligned}\frac{d}{dx}x =& 1\\ \frac{d}{dx}e^{\ln x} =& 1\\ \text{Chain rule.}\\ e^{\ln x}\frac{d}{dx}\ln x =& 1\\ x\frac{d}{dx}\ln x =& 1\\ \frac{d}{dx}\ln x =& \frac{1}{x}\end{aligned}$$

Because $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$, we can use the chain rule to find any derivative of the form $\ln(g(x))$.

$$\frac{d}{dx}\ln(x^2)$$

$$\begin{aligned}f = \ln x\ g = x^2\\ f' = \frac{1}{x}\ g' = 2x\\ \frac{d}{dx}\ln(x^2) = \frac{2x}{x^2}\end{aligned}$$

13 Applications of Derivatives

Let's say we are given a function that represents position at any time t in seconds $(s(t))$. Let's do some exploring.

$$s(t) = t^3 - 6t^2 + 9t$$

1. Find a representation of the velocity in terms of time.

   Using some intuition we could think about velocity as the change in our position. This means that the derivative of our position function would be equal to our velocity.

   $$\frac{d}{dt}s(t) = v(t) = 3t^2 - 12t + 9$$

2. Where is the velocity equal to 0?

   If we factor our velocity equation, and set it equal to 0, we can find where the velocity is 0, or when the particle is not moving.

$$(3t-3)(t-3) = 0$$
$$t = 1 \text{ and } t = 3$$

3. Where is the particle moving in a positive direction?

   We need to see where the velocity function is positive (using interval notation we find):

$$(0,1) \text{ and } (3,\infty)$$

   Notice the open parenthesis. This is because our particle is not moving at 0, 1 or 3 seconds.

4. Describe how the velocity is changing.

   Starting at $t = 0$, the particle is moving in a positive direction (left to right). At $t = 1$, the particle stops moving and turns around. It is now going in a negative direction (right to left) until $t = 6$. At this point, the particle stopped again and switched directions. It took off in a positive direction (left to right) forever.

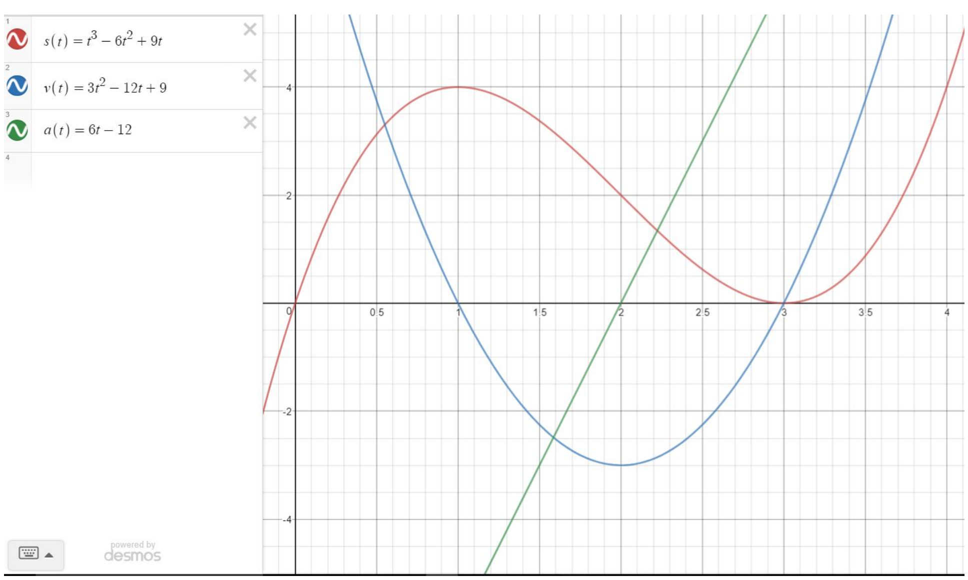
5. Find the acceleration at $t = 4$.

   We can think of acceleration as the change in our velocity which means acceleration is the derivative of velocity. Therefore,

$$\frac{d}{dx}v(t) = a(t) = 6t - 12 \Rightarrow a(4) = 12$$

.

6. Graph $s(t)$, $v(t)$, and $a(t)$.



7. When is the particle speeding up and slowing down?

   We know if we are slowing down our acceleration is negative and if we are speeding up the acceleration is positive.

   Slowing down $\Rightarrow (0,2)$

   Speeding up $\Rightarrow (2,\infty)$

The big take away here is that given a position function, we can find velocity, and given velocity, we can find acceleration.

$$\frac{d}{dt}s(t) = v(t)$$
$$\frac{d}{dt}v(t) = a(t)$$

14 Natural growth and decay.

When we look at functions that change by a constant ratio over time. Derivatives take the form

$$y(t) = k\big(y(0)e^{kt}\big)$$

where k is the rate of your growth or decay, and $y(0)$ is the initial amount, and t is time.
Look at this table. We can complete it using calculus!

Year	Population (millions)
1950	2560
1960	3040
1993	
2020	

We begin by setting up our equation

$$y(t) = y(0)e^{kt}$$

Plug in known values.

$$3040 = 2560e^{10k}$$

We can say this because the original population $y(0) = 2560$ and the population 10 years later is 3040.
From here, we can solve for k.

$$
\begin{aligned}
3040 &= 2560e^{10k} \\
&\quad \text{Divide both sides by 2560.} \\
\frac{3040}{2560} &= e^{10k} \\
&\quad \text{Take natural log of both sides (ln).} \\
\ln\left(\frac{3040}{2560}\right) &= \ln e^{10k} \\
&\quad \text{Cancel } \ln e = 1 \text{ on right side} \\
\ln\left(\frac{3040}{2560}\right) &= 10k \\
&\quad \text{Divide by 10} \\
\frac{1}{10}\ln\frac{3040}{2560} &= k
\end{aligned}
$$

Now we have

$$2560e^{\left(\frac{1}{10}\ln\frac{3040}{2560}\right)t}$$

.

From here we can fill in our table by making $t = 43$ and $t = 70$.

Year	Population (millions)
1950	2560
1960	3040
1993	5360
2020	8524

15 Related Rates.

Related rates is an application of implicit differentiation (Section 11). As the name says we want to discover how things that are changing are related. That sounds a little weird, so let me explain. If we are draining a pool (changing the volume of water in the pool), can we find out how fast the depth of the water is changing? Based on velocity, can we tell how fast two things are moving away from (or getting closer to) each other? These are things we can answer with related rates problems. Let's see how it works!

15.1 Example 1

Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Our first step is to recognize what we are looking for. In this case we are trying to find how fast the radius is changing at a specific time. In terms of Leibniz notation, we are looking for the derivative of the radius with respect to time ($\frac{dr}{dt}$). In other words, we want to know how fast the radius is changing at any time t. That means we are looking for a function $r'(t)$ or $\dfrac{dr}{dt}$ =______.

Next we need to brake down what we know.

1. How fast the volume is changing. Change in volume over time $\dfrac{dV}{dt}$=5cm^3/min

2. Diameter at the point we want to know is 20 cm. $d = 20$ $r = 10$

First we need to find an equation that connects volume and radius of a sphere. One equation we have for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Let's try taking the derivative with respect to time.

$$\frac{d}{dt}(V = \frac{4}{3}\pi r^3) \qquad \text{Using implicit differentiation we get}$$
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we have the change in volume over time, the change in radius over time, and radius as variables in our new equation.

At the beginning of the problem we decided, "We are trying to find how fast the radius is changing over time ($\frac{dr}{dt}$)." Let's solve for $\frac{dr}{dt}$.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$\frac{\frac{dV}{dt}}{4\pi r^2} = \frac{dr}{dt}$$

We know $\dfrac{dV}{dt} = 5\text{cm}^3/\text{min}$, and $r = 10$. Thus we can say,

$$\frac{5}{4\pi 10^2} = \frac{dr}{dt} = \frac{5}{400\pi} = \frac{1}{80\pi}$$

. We know $\dfrac{dr}{dt}$ is a rate so our label is cm/min, so $\dfrac{dr}{dt} = \dfrac{1}{80\pi}$cm/min.

When we solve related rates problems, we want to find or make an equation that related everything. Then, when we take the derivative with respect to time (normally), we get an equation that relates the rates of everything. We can then solve for the rate we want and plug in what we know.

15.2 Things to look for/do

When doing related rates problems, there are a few things that you can look for that will make problems easier/possible.

1. Similar triangles

2. Pythagorean's Theorem

3. Geometric formulas

4. Drawing a picture

5. Implicit differentiation

16 First and Second derivative test

Derivatives tell us information about a function. A **critical point or critical number** of a function is a minimum, maximum or a vertical asymptote. In other words, the critical points of a function occur when the first derivative of a function is 0 or undefined. The second derivative can tell us where our concavity switches (I will explain what concavity is later). These points where we switch from concave up to concave down are called **inflection points**. An inflection point can be found when the second derivative is 0. Let's look at how these tests work.

The first derivative test gives us our critical numbers. By finding where the derivative is 0 or where it doesn't exist. Let's have $f(x) = x^4 - 4x^3$

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 \\ 0 &= 4x^2(x-4) \\ x &= 0,4 \end{aligned}$$

Thus, in this case we have critical numbers at $x = 0$, and $x = 4$.

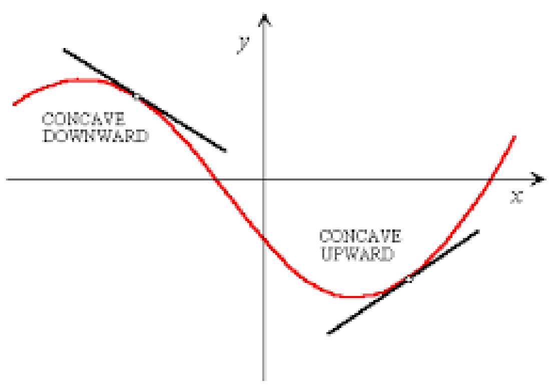
Critical numbers tell us when we have a local minimum, local maximum, where the function is flat, or a vertical asymptote. The way we can interpret our data is with a sign chart. If we think about what zeros of a function mean, we know the only time the our function output will change from positive to negative (or vis versa), is when we cross the x-axis (at a zero of the function). Combining this intuition with the zeros of our derivative, we know that our function has to either be increasing or decreasing from on each of the following segments. $(-\infty,0)$, $(0,4)$ and $(4,\infty)$. With a sign chart, we can test one number in each segment in order to see if our function is increasing **or** decreasing.

$$f'(x) = 4x^2(x-4)$$

Segment	$(-\infty,0)$	$(0,4)$	$(4,\infty)$
Test Point	$f'(-1) = -20$	$f'(1) = 100$	$f'(5) = 125$
Sign	-	-	+
Direction	$\searrow$	$\searrow$	$\nearrow$

From this table, we can see $f(x)$ has a negative slope in the segments $(-\infty, 0)$ and $(0,4)$ and a positive slope in the segment $(4, \infty)$. This means our graph is going down and then at $x = 0$, our function flattens out, and then proceeds to go down again. We have a negative slope until $x = 4$, and then our slope goes positive from then on. If we think about what this would look like, we would have a local minimum, and actually an **absolute** minimum, at $x = 4$.

Now we need to find the inflection points of the graph which we can do by setting the second derivative equal to 0. The inflection points, like I said before, are points where our function switches from concave up to concave down. Concavity is a word used to describe the shape of the graph. If our graph is **concave up**, it will be shaped like a bowl (like the graph of $y = x^2$). If our graph is **concave down**, it will be shaped like an arch (like the graph of $y = -x^2$). An inflection point is where a graph switches from one kind of concavity to another. Here is a picture.



As you can see, the tangent line is on top of the function when concave down, and underneath when concave up. The inflection point of this graph is at $x = 0$. Let's look at our function again.

$$\begin{aligned} f''(x) &= 12x(x-12) \\ 0 &= 12x(x-12) \\ x &= 0, 12 \end{aligned}$$

Thus, we have an inflection point at $x = 0$ and $x = 12$.

When we make our sign chart, a negative second derivative implies that the graph is concave down. Oppositely, a positive second derivative implies that the graph is concave up.

Segment	$(-\infty, 0)$	$(0, 12)$	$(12, \infty)$
Test Point	$f''(-1) = 156$	$f''(1) = -132$	$f''(13) = 156$
Sign	+	-	+
Shape	$\smile$	$\frown$	$\smile$

From this table we can see the concavity switches from concave up to concave down at $x = 0$, and back to concave up at $x = 12$.

16.1 Completing and Interpreting Data to Graph a Function

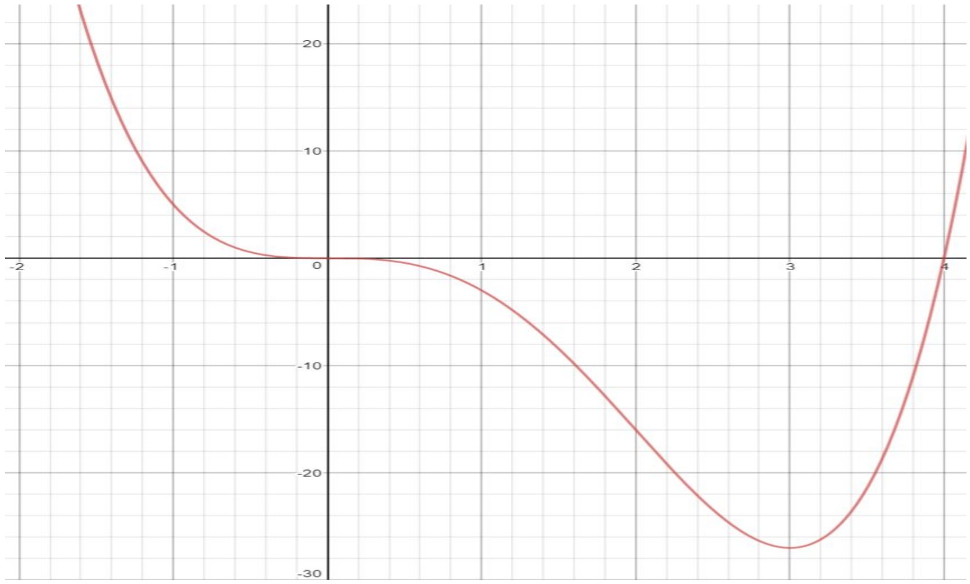
In order to make an accurate sketch of a graph we need to know what the function is doing as it goes to positive or negative infinity (the asymptotes). If you remember from section 3, this means we need to find the following limits **OF THE ORIGIONAL FUNCTION** $f(x)$. (We are still using the same function as in section 16)

$$\lim_{x\to\infty} f(x)$$
$$= \lim_{x\to\infty} x^4 - 4x^3 = \infty$$
$$\lim_{x\to-\infty} f(x)$$
$$= \lim_{x\to-\infty} x^4 - 4x^3 = \infty$$

It is also good to have x intercepts ($y = 0$) and y intercept ($x = 0$) of your function. If we do this, we get an intercept at (0,0), and (4,0).
Here is a complete list of everything we need to know what a graph looks like:

1. What are the x and y intercepts of the graph?
2. What is the domain of the graph?
3. Does the graph have any asymptotes? If yes, what are they?
4. What are the intervals of increase decrease?
5. What are the local minimum and maximum values?
6. What are the intervals of concavity?
7. What are the inflection points?

If we put all this information together, we will get a graph that looks similar to this!!!



17 L'Hospital's Rule

L'Hospital's Rule is an easier way to get around limits that are in indeterminate forms. To use L'Hospital's Rule a function **MUST** be in an indeterminate form.

There are 7 different indeterminate forms:

1. $\frac{0}{0}$
2. $\frac{\pm\infty}{\pm\infty}$
3. $0 \times \infty$
4. $\infty - \infty$
5. 0^0
6. 1^∞
7. ∞^0

L'Hospital's Rule states: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$
$$\text{or that } \lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

So what does that mean?

Basically, if you are trying to take a limit, and when you plug in your "a" you get either $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then take the derivative of the top, and the derivative of the bottom (SEPRATELY don't use quotient rule), and then compute the limit. For example,

$$\lim \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{x^2-x}{x^2-1} = \frac{0}{0}$$
$$= \lim \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{2x-1}{2x} = \frac{1}{2}$$

That is great when we are in the forms $\frac{0}{0}$ **or** $\frac{\pm\infty}{\pm\infty}$, but there are other indeterminate forms that we listed earlier. What do we do when we don't have a $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$?

I am glad you asked!!! If we are in an indeterminate form such as $0\times\infty$, $\infty-\infty$, 0^0, 1^∞, or ∞^0, believe it or not, we can use our favorite tool...ALGEBRA!!! Using algebra, we can make any of the 7 indeterminate forms into $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. I bet this is exactly the answer you were hoping for, but either way, let's do an example problem.

$$\lim_{x\to 0^+} (x^x)$$
$$= \lim_{x\to 0^+} (e^{x\ln(x)})$$

$$=e^u \qquad u = \lim_{x\to 0^+} (x\ln(x))$$
$$=e^u \qquad u = \lim_{x\to 0^+} \left(\frac{\ln(x)}{\frac{1}{x}}\right)$$
Use L'Hospital's Rule
$$=e^u \qquad u = \lim_{x\to 0^+} \left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right)$$
$$=e^u \qquad u = 0$$

$$e^0 = 1$$

In terms of the calculus, L'Hospital's Rule is not very difficult, but the algebra can get pretty harry. Practice is key for L'Hospital's Rule. Just remember that L'Hospital's rule can only be used when you are in $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

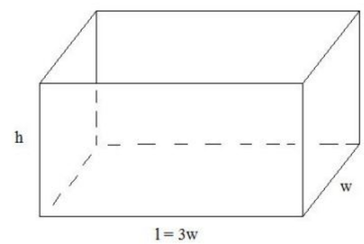
18 Optimization

Optimization is the process of figuring out how to get the most out of out of something. This would be like minimizing cost to maximize profit, making machines more efficient or maximizing volume and minimizing material. Using calculus, you might be able to see how we could do this. We know if we have a function that represents what we want to find, we can take the derivative, set it equal to 0 and solve. Where the derivative is 0, our function will have a minimum or a maximum or the optimal point.

As you might think, optimization uses more algebra than calculus. In terms of calculus, we are only taking one derivative. Other than that, we have to make a function that represents what we want, and then we have to solve for the zeros of the functions. Let's look at a problem.

We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $\$10/ft^2$ and the material used to build the sides cost $\$6/ft^2$. If the box must have a volume of $50ft^3$ determine the minimum cost to build the box.

First, let's draw a picture.



We know the volume for a box like this is $V = lwh$. We also know $l = 3w$ and the volume must be 50. This means our first equation we can make is

$$50 = 3w^2h$$

We also need to remember that the question is asking for the cost of the box so that would mean we would need a cost function (C). If we break down the cost of producing the box into the cost to produce each side of the box it will be easier.

First we know we have a top and a bottom that are the same size, and specifically they are $3w^2$. Since we have two of them our cost function needs to take that into account. This means the first term of our cost function would be $2(3w^2)$. In the cost function we obviously need cost, and the cost for the base material is $\$10/ft^2$. This means so far our cost function is

$$C = 2(3w^2)(10)$$

We know we have two different sized sides, and we have two of each of them. The first side would be wh. Remember we have two of them, and the material for the sides costs $\$6/ft^2$ this means $C = 2(wh)(6)$. Now our cost function is

$$C = 2(3w^2)(10) + 2(wh)(6)$$

The second type of side is modeled by the expression $lh = 3wh$, and like the other side, the cost for the material is $\$6/ft^2$. From this we can say $C = 2(3wh)(6)$, and our total cost function becomes

$$C = 2(3w^2)(10) + 2(wh)(6) + 2(3wh)(6)$$

Let's simplify this a bit.

$$C = 60w^2 + 36wh + 12wh$$
$$C = 60w^2 + 48wh$$

Since we want to find the minimum of the cost function, we would need to find C' and set it equal to 0. The problem is that we have two variables, and that would make this a mess. Do you remember the first equation we made? $50 = 3w^2h$. If we solve this for h, we can substitute into our cost function.

$$h = \frac{50}{3w^2}$$

If we substitute this into our cost function we get

$$C = 60w^2 + 48w\left(\frac{50}{3w^2}\right)$$

Simplifying this we get

$$C = 60w^2 + \frac{800}{w}$$

Now if we find C' we get

$$C' = 120w - \frac{800}{w^2}$$

At this point we want to find the zeros of the derivative
Using some algebra, we can say

$$C' = \frac{120w^3 - 800}{w^2}$$

If we let the numerator equal 0 we can find the zeros. Therefore we get $w = \sqrt[3]{\frac{800}{12}} \approx 1.8821$ and $w = 0$. Obviously in this problem the width can't equal 0 so we can throw that out. Now we know $w \approx 1.8821$.

Remember we want the cost, so we can say

$$C(1.8821) \approx \$637.60$$

As you can see, the calculus is the easy part of this problem. Remember to keep in mind what you want to solve for. In this problem, we wanted cost not the dimensions of the box.

19 Antiderivatives

Just like in elementary school when we all learned subtraction was the inverse of addition, or division was the opposite of multiplication, in Calculus, we have derivatives, but what if we want to go the opposite way? As we know derivatives follow the general form $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$, but what if we started with $f'(x) = nx^{n-1}$, how can we find $f(x)$? **Can** we find $f(x)$? The answer is YES! We can find $f(x)$, but it isn't "perfect." We call this "reverse" version **antiderivatives**.

To find $f(x)$, we start with $f'(x) = nx^{n-1}$. Well, what do we do when we take a derivative?

1. Multiply by the function by the exponent
2. Subtract 1 from the exponent

What would be the opposite of this? (Remember when you solve an equation like $x^2 + 3 = 7$ you reverse your operations, thus, you would subtract first.)

1. Add 1 to your exponent
2. Divide by the new exponent (or multiply by the reciprocal of the new exponent)

Therefore, to find $f(x)$, if $f'(x) = nx^{x-1}$, we add one to the exponent, and thus get nx^n, and then we divide by the new exponent (or multiply by the reciprocal of the new exponent). By doing that, we get $f(x) = \frac{1}{n}nx^n = x^n$.

Now let's use numbers!!!

$$f'(x) = 7x^6 \Rightarrow f(x) = \tfrac{1}{7}7x^7 = x^7$$

$$f'(x) = 24x^9 \Rightarrow f(x) = \tfrac{1}{10}24x^{10} = \tfrac{24}{10}x^{10}$$

Do you remember when I said, "We can find $f(x)$, but it isn't perfect"? Let me give you a general idea of what I meant by that. What happens when we know $f(x) = x^2 + 1$? Well, we know $f'(x) = 2x$. Now let's go backwards. If we are given $f'(x) = 2x$, we can solve for the antiderivative and say $f(x) = x^2$, **but it doesn't**; we know that. We know $f(x) = x^2 + 1$. This is were our antiderivatives struggle. It doesn't matter if $f(x) = x^2 + 1$, or if $f(x) = x^2 + 100$, our derivative will be $f'(x) = 2x$, and thus our antiderivative will be x^2. How can we fix this problem?

What if when we find our antiderivative, we do this: $f'(x) = nx^{n-1}$; $f(x) = \frac{1}{n}nx^n + C$ where C is just some random number we don't know (an arbitrary constant). Let's go back to our $f(x) = x^2 + 1$ example.

As we know, $f'(x) = 2x$, and our antiderivative is $f(x) = x^2$. Now if we take our antiderivative, and add C we get $f(x) = x^2 + C$. Now if we pick the "right" number for C, we get $f(x) = x^2 + 1$.

HOLD UP THOUGH!!!!!

what about $f(x) = x^2 + 100$? Does our antiderivative, "$x^2 + C$," work? YOU BET IT WILL!!! All we have to do is say $C = 100$.

I know math can get a little boring, but why do we just have to add C? We don't!!! We can add anything we want thus, "x^2+  " is perfectly acceptable as long as it is clear that  is some arbitrary constant.

Now, if we keep going, and take the antiderivative of our antiderivative ($x^2 + C$), we get $\frac{1}{3}x^3 + \frac{1}{1}Cx + D$. This comes from the fact that $C = Cx^0$. We still need our arbitrary constant, but C was already used, thus we used D.

Sometimes, we have the information to solve for C (or D,E,...). Say we are given $f'(x) = 1 + 3\sqrt{x}$ and $f(1) = 0$. We have enough information to solve for C!

$$\begin{aligned} f'(x) =& 1+3\sqrt{x} = 1+3x^{\frac{1}{2}} \\ f(x) =& x+\frac{3}{2}3x^{\frac{3}{2}}+C \\ \text{general antiderivative} \Rightarrow f(x) =& x+\frac{9}{2}x^{\frac{3}{2}}+C \\ \text{Because } f(1)=0 \Rightarrow 0 =& 1+\frac{9}{2}(1^{\frac{3}{2}})+C \\ C =& -\frac{11}{2} \end{aligned}$$

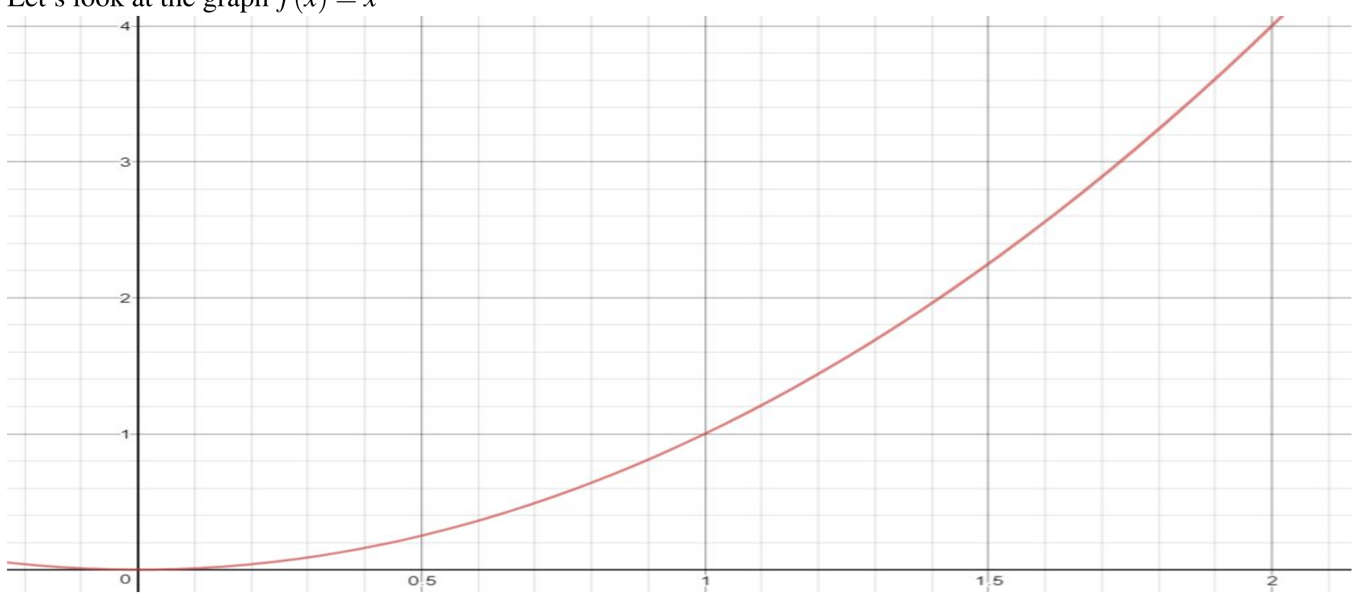
$$f(x) = x+\frac{9}{2}x^{\frac{3}{2}}-\frac{11}{2}$$

Antiderivatives also work for trigonometry. Just like the derivative of $\sin x = \cos x$, the antiderivative of $\cos x = \sin x$.

20 Area under a curve and Riemann Sums

Have you ever thought about the area under a graph? I hadn't before I took calculus. However, some people did, and they explored and discovered some really cool things. Area is a very fundamental concept in math. You first discovered area in elementary school, and one of the first formulas you learned was the area of a rectangle ($A = lw$). As it turns out rectangles work well to give us a rough estimate of the area under a curve. Let's look at how that works.

Let's look at the graph $f(x) = x^2$



First off, we have to know how far to go. Let's start on the interval [0,2]. If we are going to add up the area of rectangles, we should probably decide how many rectangles we want to use. We should start with 2. As we know, $A_{rect} = lw$ this means we need a length and a width. How can we define these dimensions?

The width is easy. We just need to divide the length of our interval by the number of rectangles we want to use. This means on the interval $[a,b]$, $w = \frac{b-a}{n}$ where n is the number of rectangles we want. We like to call the width Δx in calculus because it is the change on the x-axis. This means

$$\Delta x = \frac{b-a}{n}$$

The length of our rectangles (since we are going up and down let's call it height) can be defined in a few ways.

Assume $f(x)$ is a function and we are on $[a,b]$

1. **Left-endpoint Rule**. We can use what are called left-endpoints to get the heights of our rectangles. If we think about the first rectangle, our x-coordinates will be at $x = a$ and $x = a + \Delta x$. If we base the height off the leftmost point of the rectangle, the height of the first rectangle would be $f(a)$.

   Similarly, the second rectangle would be at x-coordinates $x = a + \Delta x$ and $x = a + 2\Delta x$. On this rectangle our left-endpoint rule tells us our height is $f(a + \Delta x)$.

2. **Right-endpoint Rule**. We can use what are called right-endpoints to get the heights of our rectangles. If we think about the first rectangle, our x-coordinates will be at $x = a$ and $x = a + \Delta x$. If we base the height off the rightmost point of the rectangle, the height of the first rectangle would be $f(a + \Delta x)$.

   Similarly, the second rectangle would be at x-coordinates $x = a + \Delta x$ and $x = a + 2\Delta x$. On this rectangle our right-endpoint rule tells us our height is $f(a + 2\Delta x)$.

3. **Midpoint Rule**. We can use what are called midpoints to get the heights of our rectangles. If we think about the first rectangle, our x-coordinates will be at $x = a$ and $x = a + \Delta x$. We can find the midpoint using this formula: $m = \frac{(a+\Delta x)-a}{2} = \frac{\Delta x}{2}$ If we base the height off the midpoint of the rectangle, the height of the first rectangle would be $f(\frac{\Delta x}{2})$.

   Similarly, the second rectangle would be at x-coordinates $x = a + \Delta x$ and $x = a + 2\Delta x$. To find the midpoint of this rectangle do the following: $m = (a + \Delta x) + \frac{\Delta x}{2}$. On this rectangle our midpoint rule tells us our height is $f((a + \Delta x) + \frac{\Delta x}{2})$.

From these rules, let's redefine our definition of the area of the rectangles to be $A_{rect} = \Delta x f(c)$ where c is the left or right-endpoint or the midpoint of the rectangle. Using this definition of area, we can say the area under the curve or the **Riemann Sum** will follow the form

$$A = \Delta x f(c_1) + \Delta x f(c_2) + \cdots + \Delta x f(c_n)$$

and if we factor out the Δx from each term we get

$$A = \Delta x\Big(f(c_1) + f(c_2) + \cdots + f(c_n)\Big)$$

Let's try doing our problem with each rule.

1. $[2,0]$

2. 2 rectangles

3. Left-endpoints

4. $\Delta x = \frac{2-0}{2} = 1$

5. $c_1 = 0, c_2 = 1, f(c_1) = 0, f(c_2) = 1$

6. 1(0+1)=1

The awesome thing about adding up rectangles is that the more rectangles we have, the better our estimate is. Another great things is that this process is linear in the sense that we do everything the same way each time.
Since we are just adding up terms based on a given process, we can use summation notation.

$$\sum_{i=0}^{n} f(c_i)\Delta x$$

Applying this to $f(x) = x^2$ for 10 rectangles on the interval [0,2], using left-endpoints, we do this:
$\Delta x = \frac{2}{10}$, $c_i = i\Delta x$ where $f(c_i)$ is the height of the rectangle.

$$\sum_{i=0}^{9} \left(\left(\frac{2i}{10} \right)^2 \left(\frac{2}{10} \right) \right) = 2.28$$

Taking this one step further, we can take a limit as n approaches infinity.

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(c_i)\Delta x$$

This will make the rectangles infinitely thin making our approximation the exact answer!! To do this with $f(x) = x^2$, $[0,2]$ we would do the following:

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left(\left(\frac{2i}{n} \right)^2 \left(\frac{2}{n} \right) \right)$$

For the purposes of this class you will not actually need to solve this limit, but you will need to know how to set it up.

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left(\left(\frac{2i}{n} \right)^2 \left(\frac{2}{n} \right) \right) = \int_0^2 x^2 dx = 2\frac{2}{3}$$

The only difference between left-endpoints and right endpoints is that we will start i at 1 instead of 0 and end at $i = 10$ instead of 9. This means the height of our first rectangle will be $f(\Delta x)$ instead of $f(0)$. Likewise, the height of our last rectangle will be $f(\Delta x \cdot 10) = f(2)$ instead of $f(\Delta x \cdot 9)$ or $f(1.8)$.

So by using right-endpoints we get

$$\sum_{i=1}^{10} \left(\left(\frac{2i}{10} \right)^2 \left(\frac{2}{10} \right) \right) = 3.08$$

Taking this one step further, we can take a limit as n approaches infinity.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

This will make the rectangles infinitely thin making our approximation the exact answer!! To do this with $f(x) = x^2$, $[0,2]$ we would do the following:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)\right)$$

For the purposes of this class you will not actually need to solve this limit, but you will need to know how to set it up.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)\right) = \int_0^2 x^2 dx = 2\frac{2}{3}$$

Midpoints are a little more tricky to find a formula for c_i. Using the same guides as before ([0,2], $f(x) = x^2$), c_i will be the following:

$$c_i = (i-1)\Delta x + \frac{\Delta x}{2} = \frac{2(i-1)}{10} + \frac{2}{20}$$

Thus we have

$$\sum_{i=0}^{9}\left(\frac{2(i-1)}{10} + \frac{2}{20}\right)^2 \frac{2}{10} = 2.66$$

Taking this one step further, we can take a limit as n approaches infinity.

$$\lim_{n\to\infty}\sum_{i=0}^{n} f(c_i)\Delta x$$

This will make the rectangles infinitely thin making our approximation the exact answer!! To do this with $f(x) = x^2$, $[0,2]$ we would do the following:

$$\lim_{n\to\infty}\sum_{i=0}^{n}\left(\left(\frac{2(i-1)}{n} + \frac{2}{2n}\right)^2\left(\frac{2}{n}\right)\right)$$

For the purposes of this class you will not actually need to solve this limit, but you will need to know how to set it up.

$$\lim_{n\to\infty}\sum_{i=0}^{n}\left(\left(\frac{2(i-1)}{n} + \frac{2}{2n}\right)^2\left(\frac{2}{n}\right)\right) = \int_0^2 x^2 dx = 2\frac{2}{3}$$

As you look through this you are seeing a big stretched out “S”. This is the symbol for an integral. When you take an integral (which we will talk about more later), you are finding the limit as n goes to infinity of our Riemann SUM...thus the stretched out “S” stands for sum.

21 Definite integrals and the Fundamental Theorem of Calculus

If you remember in the last section, we took the limit as the width of the rectangles went to 0, and this turned our sum of areas into an integral. We specifically had $\int_0^2 x^2 dx = 2\frac{2}{3}$. How does that work? What does that mean?

In calculus we have a lot of theorems, but the most important one is the “Fundamental Theorem of Calculus” (FTC). It kinda sounds important doesn’t it? The FTC has two parts; let’s look at part 1.

FTC Pt. 1:

If f is a continuous function on $[a,b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

With this theorem, we can evaluate $\int\limits_0^2 x^2 dx$. First, we need an antiderivative of $f(x) = x^2$ which is $F(x) = \frac{x^3}{3} + C$. Now we evaluate $F(b) - F(a)$. This is notated as $\frac{x^3}{3} + C\Big|_0^2$. $F(2) - F(0) = \frac{8}{3} + C - \left(\frac{0}{3} + C\right) = \frac{8}{3} = 2\frac{2}{3}$. Therefore,

$$\int_0^2 x^2 dx = 2\frac{2}{3}$$

Did you notice how the $+C$ and the $-C$ canceled out? That's why the theorem says "$F(x)$ is ANY antiderivative of $f(x)$."

Now that we understand part 1 of the FTC, we can move on to part 2.

FTC Pt. 2:

If f is a continuous function on the open interval I containing the point a, then the function $\int\limits_a^x f(t)dt$ is differentiable on I and for all x in I,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

I know that probably went over your head, but that is totally okay. Basically, all this theorem is saying is that the derivative of the integral is the function, or that they are opposite operations just like multiplication and division. That is huge though because that connects derivatives and integrals. If we do one with numbers, it will look like this:

$$\frac{d}{dx}\int_a^x t^2 + 4t + 5\ dt = x^2 + 4x + 5$$

22 U-substitution

When taking integrals, u-substitution gives us the ability to undo chain rule. When we have an integral in the form $\int f'(x)g(f(x))dx$ we can choose a $u = f(x)$ so $\frac{du}{dx} = f'(x)$ (derivative of u with respect to x). Then we can use $\dfrac{du}{dx}$ as a fraction and get $\dfrac{du}{f'(x)} = dx$. At this point we substitute and use some algebra to cancel, and we get $\int g(u)du$, and this is an integral we can do.

So what does that mean with numbers??? ◡̈

$$\int_0^1 x\cos(x^2)dx$$

Let $u = x^2$, this means $\frac{du}{dx} = 2x$ which means $\dfrac{du}{2x} = dx$

Thus, we can plug in our variables and say

$$\int_0^2 x\cos(u)\frac{du}{2x}$$

$$= \frac{1}{2}\int_0^2 \cos(u)du$$

There is something different here. The bounds of our integral changed. Why is that?

The bounds changed because our variable changed. u literally is $2x^2$. This means that we need to change our bounds according to our variable u. Thus for our new bounds we will plug in our old bounds as x in $u(x)$. This means $u(0) = (0)^2 = 0$ and $u(1) = 2(1)^2 = 2$, or 0 and 2.

$$\int_0^2 \cos(u) = \sin(u)\Big|_0^2 = \cos(1) - \cos(0) = \cos(1) - 1$$

Let's try one more!!!

$$\int_0^1 \frac{3x^2}{(x^3+3)^2}dx$$

Let $u = x^3 + 3$, and thus $\frac{du}{dx} = 3x^2$, and $\dfrac{du}{3x^2} = dx$

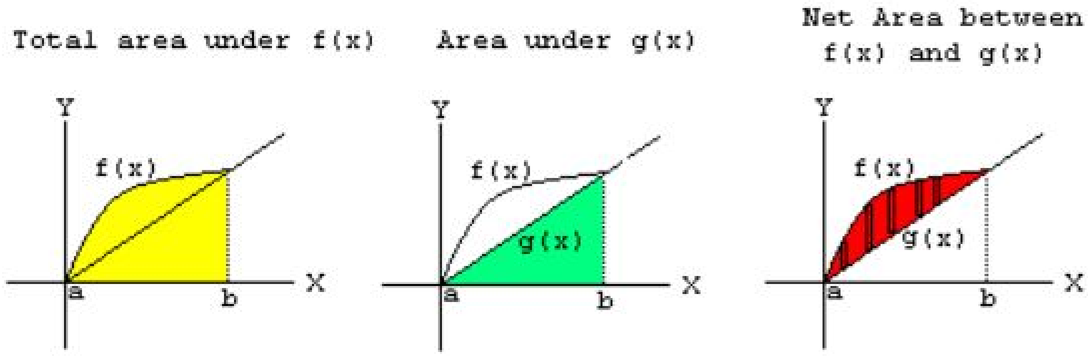
Thus we can say (notice the bounds):

$$\int_3^4 \left(\frac{3x^2}{u^2}\right)\left(\frac{du}{3x^2}\right)$$

$$= \int_3^4 \frac{1}{u^2}du = -\frac{1}{u}\Big|_3^4 = -\frac{1}{4} + \frac{1}{3} = \frac{1}{12}$$

23 Area between curves

Let's say we have two functions $f(x)$ and $g(x)$, and we want to find the area between the two functions. as we know by intuition, if we take the area under the upper graph (the integral), and subtract the area under the lower graph (the integral), we can find the area between the two graphs.



In this picture, there is an a and b. This makes sense because when we integrate, we need to integrate from a to b. How do we find a and b though? If we look at the picture, a and b are the points where the graphs cross, or in mathematical terms a and b are the x-coordinates when $f(x) = g(x)$.

As we know we need to subtract the lower function from the upper function, but how do we know which one is the upper, and which one is the lower function? Well, between a and b one of our graphs will always be on top and one will always be on bottom. This means if we plug in a number c, where c is between a and b, then either $f(c) > g(c)$ or $g(c) > f(c)$, and this will be true on the entire interval. Let's try a problem!

$$f(x) = \sqrt{x},\, g(x) = x^2$$

First find a and b.

$$\begin{aligned} \sqrt{x} &= x^2 \\ x &= x^4 \\ 0 &= x^4 - x \\ x &= 0, 1 \end{aligned}$$

Thus, $a = 0$ and $b = 1$
Let's say $c = .5$ because $a < .5 < 1$

$$f(c) \approx .707,\, g(c) = .25 \Rightarrow f(x) > g(x)$$

Therefore, we can say the area between the curves is

$$\int_0^1 \left((\sqrt{x}) - (x^2)\right) dx = \left(\left(\frac{2}{3}x^{\frac{3}{2}}\right) - \left(\frac{1}{3}x^3\right)\right)\Bigg|_0^1 = \frac{1}{3}$$

24 Volumes of Revolution

24.1 The Big Idea

The idea behind volumes of revolution is fairly simple. Have you ever taken a sparkler on the 4^{th} of July, and tried to write your name with the streaks of light? Well, that is exactly what we are doing here! We have a function $f(x)$, let's say $f(x) = x$ to make things simple, and let's cut it off at $x = 0$ and $x = 1$. From here, were are just going to spin our function really fast around the x-axis so that it leaves a streak of itself behind.

What is the shape of the function's streak?

If you think about it the streak will be in the shape of a cone with a height of 1, and a radius of 1.

How can we find the volume of this shape?

Well, we know that the volume of a cone is $\frac{1}{3}\pi r^2 h$, and we know our radius and height are both 1, so we can say $V = \frac{1}{3}\pi$.

What happens if we have a more complicated function that doesn't give us a "happy" shape like a cone? Well whatever shape we have, we can slice it into what we like to call "disks." A disk is a slice straight slice through our shape, and each slice will have a uniform thickness which we will label Δx. These disks will make right cylinders of which, we can find the volume by using the volume of a cylinder formula which is $V = \pi r^2 h$, and we know our $r = f(x)$, and $h = \Delta x$, thus our volume is $V = \pi f(x)^2 \Delta x$. Once we add all the volumes of the cylinders together, we can approximate our volume.

In calculus, we always want better and better approximations to the point where they are perfect. We normally do this by using a limit as something that we can control goes to 0. In this case, we can control Δx, so let's write a limit.

$$V = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi f(x_i)^2 \Delta x = \int_a^b A(x)dx$$

We know these are the same thing because the area of a circle is $A = \pi r^2$, and by substitution, we can say $A = \pi f(x)^2$. All in all, the equation you need to know is

$$\int_a^b \pi f(x)^2 dx$$

Where a and b are the bounds, and in our case they were 0 and 1.

Let's finish our problem the calculus way. If we use the formula, we get the following:

$$\int_0^1 \pi x^2 dx$$

$$=\pi \int_0^1 x^2 dx$$

$$=\pi \left(\frac{1}{3}x^3\Big|_0^1\right)$$

$$=\pi \left(\frac{1}{3}-0\right)$$

$$=\frac{1}{3}\pi$$

24.2 What if we have 2 functions?

The point of the first section was the fact that we have to find a function that represents the area of a cross-sectional slice of the solid. In the first example, we had a solid "disk-shaped" cross sectional area. However, if we have 2 functions, $g(x) = x^2$ and $f(x) = x$, and spin them around the x-axis, then our cross-sectional area will be in the shape of a donut, or what we like to call a "washer." The shape of this cross-sectional area has an outer radius, an inner radius, and a hole in the middle. To find the area of the cross-sectional slice, we take the area of the outer function, and subtract the area created by the inner radius.

If we look at $g(x)$ and $f(x)$, we can see that $g(x)$ is inner radius and $f(x)$ is the outer. This means our integral will be the following:

$$\int_a^b \pi((x)^2-(x^2)^2)dx$$

Now to find a and b, we set $g(x) = f(x)$, and solve for x. In this case we get $a = 0$, and $b = 1$.

$$\int_0^1 \pi((x)^2-(x^2)^2)dx = \frac{2\pi}{15}$$

***Note that we are talking about are not distance so both $g(x)$ and $f(x)$ must be squared and multiplied by π. Thus $\int_a^b \pi(f(x)-g(x))^2dx$ will not give you the right answer.**

24.3 What if we don't spin it around the x-axis?

So far we have found out that all we have to do is write an equation for the area of the area of a cross-section, and take the integral of it. Well, if we aren't spinning around the x-axis, we just have to write a different equation.

Let's say $f(x) = x$ and $g(x) = x^2$, and we will spin it around the line $y = -1$. The only change to our problem is the fact that the radius increased by 1. Therefore, our equation will be

$$\int \pi((1\text{+outside})^2\text{-}(1\text{+inside})^2)dx.$$

We know our a and b are still 0 and 1, thus,

$$V = \int_0^1 \pi((1+x)^2 - (1+x^2)^2)dx = \frac{7\pi}{15}$$

24.4 Practice Problems

Find the volumes of the solids created by the given bounds. Calculate based on revolutions around the x and y-axis, and about $x = -1$, and $x = 1$.

1. $f(x) = \frac{2}{7}x + 5$, from $x = 0$ to $x = 3$
2. $f(x) = \sqrt{x}$, from $x = 0$ to $x = 4$
3. $f(x) = x^{-3}$, from $x = 1$ to $x = 3$
4. $f(x) = x^3$, $g(x) = x^2$
5. $f(x) = 0$, $g(x) = x - 2$, $h(x) = \sqrt{x}$

25 Work

25.1 What's the Big Deal?

You have probably all covered work in a high school physics class at some point, and you know the equation for it is Work=force $\times$ distance, and Force=mass $\times$ acceleration, and therefore $W = mad$. In high school that works great, but in real life there are more things going on. Say you are pulling a bucket out of a well with a rope. You know that the bucket has mass, but what about the rope? The rope has mass too! The thing about the rope is that every time you pull it, you have to lift less and less of the rope the higher you lift the bucket. Now you can see how calculus might be involved in work.

If you think about it, force will become a function of the mass, and your mass will vary depending on the distance traveled. Let's put some numbers on this now. Let's say the rope has a mass of 0.5 kg/m, the bucket has a mass of 3 kg, and the bucket is 10 m down. With these numbers, we can write an equation for force.

$$\text{Force } f(x) = 9.8(.5(10 - x) + 3)$$

where x is the length of the rope, and 9.8 m/s^2 is acceleration due to gravity.
We know $W = fd$, and $d = \Delta x$ so we can say

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)xdx$$

We let a and b be our bounds, and in this case they will be 10 and 0, but how do we know which is a and which is b?
As we know, the mass of our rope is going down the more we lift the bucket, and our mass equation has a negative slope, so we know we need to let out $a = 0$, and our $b = 10$. Therefore,

$$\int_0^{10} (49 - 4.9x + 29.4)xdx = \frac{31850}{3} \text{ joules}$$

25.2 Practice Problems

1. A force of, $F(x) = x^2 cos(3x) + 2$, x is in meters, acts on an object. What is the work required to move the object from $x = 3$ to $x = 7$?

2. A spring has a natural length of 18 inches and a force of 20 lbs is required to stretch and hold the spring to a length of 24 inches. What is the work required to stretch the spring from a length of 21 inches to a length of 26 inches?

3. A cable that weighs $\frac{1}{2}$ kg/meter is lifting a load of 150 kg that is initially at the bottom of a 50 meter shaft. How much work is required to lift the load $\frac{1}{4}$ of the way up the shaft?

26 Integration by Parts

26.1 The Formula

Integration by parts is another option you now have for turning an impossible integral into something you can do. For a basic example, you can't find $\int xe^x dx$. In general, it is really hard to find the antiderivative of something in the form of $\int u\frac{dv}{dx}dx$. Any time we see something in this form, we will use a method called integration by parts.

I am going to get straight to the point. The equation you need to know for integration by parts is the following:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Any time you do integration by parts, you will have to do these steps.

1. Choose a u and a $\frac{dv}{dx}$. Keep in mind that in the end you want a simpler integral so
   $\int v\frac{du}{dx}dx$ is simpler than $\int u\frac{dv}{dx}dx$

2. From your u and $\frac{dv}{dx}$ find a $\frac{du}{dx}$ and a v respectively.

3. Use the formula $uv - \int v\frac{du}{dx}dx$

For our problem:

$$\int xe^x dx$$

1. $u = x$, $\frac{dv}{dx} = e^x \Rightarrow$ I chose these because I know when I find u', 1 is simpler than x.
2. $\frac{du}{dx} = 1$, $v = e^x$
3. $xe^x - \int e^x(1)dx = xe^x - e^x + C$ or for any of you in Calc 1 last semester, $xe^x - e^x +$ 

That's literally all there is to it so go have fun on some practice problems.

27 Trigonometric Integrals

27.1 What is it?

Trigonometric integrals (trig ints) is the idea that you can use trig identities to re-write impossible integrals into manageable functions. Trig identities are equations such as $\sin^2 x + \cos^2 x = 1$, that have been proven to be true. In order to be able to do trig sub, you need to know the identities you can use. The following is a non-extensive list, but it will get you started.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \sec^2\theta = \tan^2\theta + 1$$
$$\csc^2\theta = 1 + \cot^2\theta$$

Using these identities, you will be able to re-write a crazy integral like

$$\int \sin^2(x)dx = \frac{2x - \sin(2x)}{4} + C$$

Let's try to do that!

27.2 How to do it.

27.2.1 Practice #1

Let's solve the integral I showed above.

$$\int \sin^2(x)dx$$

Just like integration by parts, our goal is to make things simpler. One thing we can do in this integral is try to get rid of the exponent on the sine function. On of the identities that would work to do this is $\sin^2 x = \frac{1-\cos 2x}{2}$. If we use this we get the following.

$$\int \frac{1-\cos 2x}{2} dx$$

Now we can split this up using algebra and linearity.

$$\frac{1}{2}\left(\int 1 dx - \int \cos(2x) dx\right)$$

These two integrals can now be solved with 2 methods we already know! Power rule and u-substitution. I am only going to show the u-substitution.

$$\begin{aligned}
&\int \cos(2x) dx \\
&u = 2x \qquad du = \frac{1}{2} dx \\
&\frac{1}{2}\int \cos(u) du \\
=&\frac{1}{2}\sin u \\
=&\frac{\sin(2x)}{2}
\end{aligned}$$

Finishing the other parts of the integral, we get

$$\frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right)$$

$$= \frac{2x - \sin 2x}{4}$$

There we have it! that is the same answer we had before!

27.2.2 Practice #2

Let's solve another one.

$$\int \tan^6 x \sec^4 x \ \ dx$$

Remember we want to make this simpler, and we want to use our trig identities. One way we could do that is split it into different factors. For example, $\tan^6 x \sec^2 x \sec^2 x$. From here, we could use $\sec^2 x = \tan^2 x + 1$ to replace one of the $\sec^2 x$ terms.

$$\begin{aligned}
&= \int \tan^6 x \sec^2 x \sec^2 x \ \ dx \\
&= \int \tan^6 x (1 + \tan^2 x) \sec^2 x \ \ dx
\end{aligned}$$

From here, you may be able to recognize that we can use a u-substitution if we let $u = \tan x$. This means $du = \sec^2 x \ \ dx$.

$$
\begin{aligned}
&= \int u^6(1+u^2) \ \ du \\
&= \int (u^6+u^8) \ \ du \\
&= \frac{1}{7}u^7 + \frac{1}{9}u^9 + C \\
&= \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C
\end{aligned}
$$

27.3 Notes

1. In your book on pg. 481, there is a strategy to solve $\int \sin^m x \cos^m x \ \ dx$.
2. On pg. 482 there are strategies for $\int \tan^m x \sec^m x \ \ dx$
3. There is a few antiderivative formulas:
   (a) $\int \tan x \ \ dx = \ln|\sec x| + C$

   (b) $\int \sec x \ \ dx = \ln|\sec x + \tan x| + C$
4. There are 3 more identities for the following:
   (a) $\int \sin Ax \cos Bx \ \ dx = \int \frac{1}{2}[\sin(Ax - Bx) + \sin(Ax - Bx)] \ \ dx$
   (b) $\int \sin Ax \sin Bx \ \ dx = \int \frac{1}{2}[\cos(Ax - Bx) - \cos(Ax + Bx)] \ \ dx$
   (c) $\int \cos Ax \cos Bx \ \ dx = \int \frac{1}{2}[\cos(Ax - Bx) + \cos(Ax + Bx)] \ \ dx$
5. Overall, trig ints are not the easiest thing, but with practice, and recognizing patterns, you will be successful.

27.4 Practice Problems

Find the integral.

1. $\int \sin^3 x \cos^4 x \ \ dx$
2. $\int \sin^5 x \ \ dx$
3. $\int \sin^2 x \cos^4 x \ \ dx$
4. $\int \sin^3 x \cos^5 x$
5. $\int \cos^2 x \tan x \ \ dx$
6. $\int e^x \sec^3 e^x \ \ dx$

28 Trigonometric Substitution

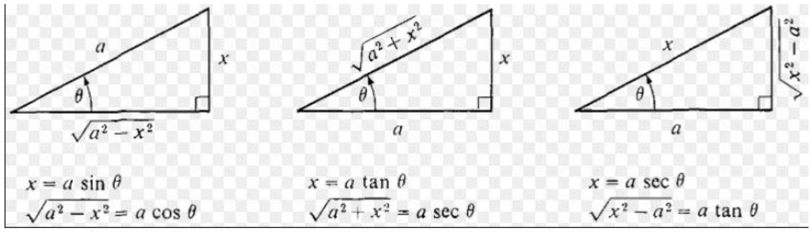
28.1 The Big Idea

Trigonometric substitution (trig sub) is a very useful method to solve integrals. If we think of a right triangle, and how we use Pythagorean's Theorem, we can see that we can get the following forms.

1. $x^2 + b^2 = a^2 \Rightarrow \sqrt{a^2 - x^2} = b$

2. $a^2+x^2=c^2 \Rightarrow \sqrt{a^2+x^2}=c$

3. $b^2+a^2=x^2 \Rightarrow \sqrt{x^2-a^2}=b$



If use see	use the sub	so that	and
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$dx=a\cos\theta\,d\theta$	$\sqrt{a^2-x^2}=a\cos\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$dx=a\sec^2\theta\,d\theta$	$\sqrt{a^2+x^2}=a\sec\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$dx=a\sec\theta\tan\theta\,d\theta$	$\sqrt{x^2-a^2}=a\tan\theta$

I am not going to show the algebra that goes into all of these, but the big idea is there. Just like the trig integrals where we were using some identities, these will work in the same way. It is really hard, and actually impossible to find $\int \sqrt{9+x^2}\ \ dx$, but using the above substitutions, we can!

28.2 Let's try it!

28.2.1 Practice #1

Let's solve the integral we looked at above.

$$\int \sqrt{9+x^2}\ \ dx$$

Using the substitution for x we have above, we get

$$= \int \sqrt{9+(3\tan x)^2}dx$$

Now we have a problem, we have a dx, and we need a $d\theta$. By our table, we know $dx = a\sec^2\theta\ \ d\theta$. This means we can say

$$= \int 3\sec^2\theta\sqrt{9+9\tan^2\theta}\ \ d\theta$$

If we look inside the radical, $9+9\tan^2\theta = 9(1+\tan^2\theta$. That looks an awful lot lie the identity we have for $\sec^2\theta$ so let's make that substitution.

$$= \int 3\sec^2\theta\sqrt{9\sec^2\theta}\ \ d\theta$$

We now have some obvious algebra.

$$= 9\int \sec^3\theta\ \ d\theta$$

You guys can do this integral. At the end, if you substitute to return back to x's, we get

$$\frac{9\ln\left(\left|\sqrt{x^2+9}+x\right|\right)+x\sqrt{x^2+9}}{2}+C$$

29 Numerical Integration

29.1 Don't freak out yet!

I know numerical integration sounds terrible, and especially after what you poor humans have been going through in this class, but I promise it isn't. Do you remember when we first started looking at integrals, and we used Riemann sums and rectangles to find the area under a curve? We had the left, right and midpoint rules, right? Well, numerical integration is the same idea, except we are going to use trapezoids. This is called the Trapezoid Rule.

If we are going to add up the area of trapezoids, we should probably know the equation for the area of a trapezoid.

$$A_{trap} = \frac{B_1 + B_2}{2}h$$

Where B_1 and B_2 are the bases of the trapezoid and h is the height (in our cases, $h = \Delta x$, and B_1 and B_2 will be $f(x_n)$ and $f(x_{n+1})$). Remember from calc 1 that $\Delta x = \dfrac{b-a}{N}$ where N is the number of trapezoids we have. Let's do a simple example!

29.1.1 Example

Let $f(x) = x^2$
On the interval $[0, 10]$

a) $N = 5$

Our first step should be identifying Δx.

$\Delta x = \dfrac{10-0}{5} = 2$, and thus $x_i = 0, 2, 4, 6, 8, 10$

Let's set up what our formula will look like quick.

$$T_5 = \frac{1}{2}\Delta x(f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + f(x_3) + f(x_4))$$

or we could say

$$T_5 = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$

This formula will work for all intervals of N trapezoids and values of x_i.

$$T_N = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + \cdots + 2f(x_{N-1}) + f(x_N))$$

Using this formula we get the following:

$$T_5 = \frac{1}{2}(2)\Big(0 + 2(4) + 2(16) + 2(36) + 2(64) + 100\Big) = 340$$

b) $N = 20$

At this point, 20 values is going to take a lot of calculations, and that just isn't fun (I thought 5 was bad enough). Let's set up a summation ($\sum$) to make this easier.

The formula for this summation is going to be the following:

$$\frac{1}{2}\Delta x\left(f(x_0)+2\sum_{i=1}^{N-1}\left(f(\Delta x\cdot i\right)+f(x_N)\right)$$

If we use this formula, we will get this:

$$\frac{1}{2}\cdot\frac{1}{2}\left(0^2+2\sum_{i=1}^{19}\left(\left(\frac{i}{2}\right)^2\right)+10^2\right)=\frac{2670}{8}=333.75$$

c) $N = 1000$

$$\frac{1}{2}\cdot\frac{1}{100}\left(0^2+2\sum_{i=1}^{999}\left(\left(\frac{i}{100}\right)^2\right)+10^2\right)=333.335$$

d)

$$\int_0^{10} x^2 = 333\frac{1}{3}$$

29.1.2 Note

1. The trapezoid rule is the average of the Left-endpoint (L_N) and Right-endpoint (R_N) approximations. Thus, $T_N = \frac{1}{2}(L_N + R_N)$.
2. The error in our estimation is given by this theorem:

   If f'' exists and is continuous, find a K_2 such that $|f''(x)| \leq K_2$ for every x in $[a,b]$ (in other words, K_2 is the maximum value of $|f''(x)|$). Then,

   $$\text{Error }(T_N) \leq \frac{K_2(b-a)^3}{12N^2}$$

   So for our problem $f'' = 2$. Thus, $K_2 = 2$. This means

   $$\text{Error }(T_{1000}) \leq \frac{2(10)^3}{12(1000)^2} = 0.0001\bar{6}$$

29.2 Midpoint Rule

Just as a quick review, we also have out good old midpoint rule from last semester. The midpoint just uses rectangles, and the formula looks like this:

$$M_N = \Delta x\left(f(c_1)+f(c_2)+\cdots+f(c_{N-1})+f(c_N)\right).$$

Where c_j is the midpoint of $[x_{j-1}, x_j]$

The error in the midpoint rule is given by this theorem:

If f'' exists and is continuous, find a K_2 such that $|f''(x)| \leq K_2$ for every x in $[a,b]$ (in other words, K_2 is the maximum value of $|f''(x)|$). Then,

$$\text{Error } (T_N) \leq \frac{K_2(b-a)^3}{24N^2}$$

29.3 Simpson's Rule

Simpson's rule is another way to estimate area. Simpson's rule uses parabolas to estimate the area under the curve. The requirement for Simpson's rule is that N must be even. The formula for Simpson's rule is the following:

$$S_N = \frac{1}{3}\Delta x\big(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{N-3}) + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\big)$$

To find the error in Simpson's Rule, we use this theorem.

If $f^{(4)}$ exists and is continuous, find a K_4 such that $|f^{(4)}(x)| \leq K_4$ for every x in $[a,b]$ (in other words, K_4 is the maximum value of $|f^{(4)}(x)|$). Then,

$$\text{Error } (T_N) \leq \frac{K_4(b-a)^5}{180N^4}$$

30 Improper Integrals

Improper integrals are a special kind of integral where the function either has a point where it is undefined on the interval $[a,b]$ (i.e. hole in the graph or an asymptote), or the interval you are integrating on is one of the following: $[-\infty, b]$, $[a, \infty]$, or $[-\infty, \infty]$. Overall, indefinite integrals are no worse than any normal integral. All you have to do is split up the integral at the problem point. It will be easier to see a example problem.

30.1 Example

$$\int_1^\infty \frac{1}{x^2} dx$$

In order to take this integral, we need to look for holes and asymptotes in $\frac{1}{x^2}$ on the interval $[1, \infty]$. We know we have an asymptote as $x \to \infty$, and thus we have a problem. Watch how we can fix that problem.

$$= -\frac{1}{x}\Big|_1^b$$

$$= \left(\lim_{b\to\infty} -\frac{1}{b}\right) + 1$$

$$= 0 + 1 = 1$$

Basically what we did was take the problem point, and take the limit as the antiderivative goes to that x-value. Since the limit was 0, we can plug in 0 to find the definite integral. We call improper integrals **CONVERGENT** if we get a finite number as our answer once we take our limits. Now let's tweak our bounds.

$$\int_0^\infty \frac{1}{x^2}dx$$

We know we have an asymptote at $x = 0$, and as $x \to \infty$ we have a problem. So let's do this one.

$$= -\frac{1}{x}\Big|_0^\infty$$

$$= \left(\lim_{b\to\infty} -\frac{1}{b}\right) + \left(\lim_{x\to 0} -\frac{1}{x}\right)$$

$$= 0 - \infty = -\infty$$

Since we didn't get a finite number, we call this improper integral **DIVERGENT**.

30.2 Think about it

Instead of using number problems, I am going to give you guys a few more examples but generalized (using letters instead of numbers). Read slow, and think as you read. These ideas will save you time in the future.

1. Let c be in $[a,b]$, and F be the antiderivative of f. If $\lim\limits_{x\to c} f(x) = \pm\infty$, then

$$\int_a^b f(x)dx = \Big(F(b) - (\lim_{x\to c} F(x)\Big) + \Big(\lim_{x\to c} f(x) - F(a)\Big).$$

2. Let $a > 0$, then

$$\int_0^a \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{1-p} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}.$$

3. Let $f(x)$ and $g(x)$ be functions so that $0 \leq g(x) \leq f(x)$ for every $x \geq a$. If

   - $$\int_a^\infty f(x)dx \text{ converges, then } \int_a^\infty g(x)dx \text{ also converges.}$$
   - $$\int_a^\infty g(x)dx \text{ diverges, then } \int_a^\infty f(x)dx \text{ also diverges.}$$

31 Arc Length

Arc length is not too difficult of a concept. Like all other concepts in calculus we begin by estimating the the length by adding up a bunch of values that are easy to work with. For arc length, we start by using the distance formula ($D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$) to find how far away 2 points are on the same curve.

As we make the distance between the points closer, we will get a better and better approximation. The summation formula for this would be

$$\sum_{n=0}^{N-1} \sqrt{\Delta x^2 + \Big(f(x_{n+1}) - f(x_n)\Big)^2}$$

If you remember the mean value theorem, we can say there is a c in (a, b) so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. So using this fact, we can say

$$\sum_{n=0}^{N-1} \sqrt{\Delta x^2 + \Big(f'(c_n)\Delta x\Big)^2}.$$

Then by algebra we get

$$\sum_{n=0}^{N-1} \sqrt{\Delta x^2 + \Delta x^2 \Big(f'(c_n)\Big)^2}.$$

Then if we factor,

$$\sum_{n=0}^{N-1} \sqrt{\Delta x^2 \Big(1 + f'(c_n)\Big)^2}.$$

This can be written as

$$\sum_{n=0}^{N-1} \sqrt{1 + \Big(f'(c_n)\Big)^2}\Delta x.$$

Now if we take the limit as $\Delta x \to 0$, we get

$$\int_a^b \sqrt{1 + (f'(x))^2}dx$$

This formula is fairly straight forward to use. Generally, once you set up the integral you will want to use a calculator or computer to approximate the value.

32 Sequences

Sequences and series are the last "big" topic in calculus. If you remember, we had limits, derivatives and integrals. Some of you might have heard about sequences, or used sequences before. I personally remember in third or fourth grade, when we had to figure out what numbers came next. We would be given something like 1, 3, 5, 7, 9..., and we would have to say "11" is the next number. This is all that sequences are. In this class, we will get something like $a_n = 2n+1$, where n is in (this symbol means "in" $\in$) the integers ($\mathbb{Z} = \{1,2,3\dots\}$). This would mean $a_0 = 1$, $a_2 = 3$, $a_3 = 5$ and so on.

A list of numbers that can be created by a rule (a function) over a given index is a **sequence**. In other words, the general term a_n can be defined by the function $f(n)$ where n is in some defined set.

- Each number, a_n, is called a term.

This means in or simple example,

1. general term = $a_n = f(n) = 2n+1$
2. $n \in \mathbb{Z}$
3. $a_0 = 1$, $a_2 = 3$, $a_3 = 5$ are three terms of our sequence.

The following are also sequences.

General Term	Domain	Sequence
$a_n = 1 - \frac{1}{n}$	$n \in \mathbb{N}$	$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$
$a_n = (-1)^n n$	$n \in \{0, \mathbb{N}\}$	0, -1, 2, -3, 4,...
$b_n = \frac{364.5n^2}{n^2-4}$	$n \in \mathbb{N}, n \geq 3$	656.1, 486, 433.9, 410.1, 396.9, ...

We will also have what are called "recursive" sequences. A recursive sequence is a sequence where we use previous terms to solve for the next one. For example, the Fibonacci Sequence (Pascal's Triangle) can be defined as $a_n = a_{n-1} + a_{n-2}$ with $n > 2$. This sequence would use the last 2 terms and add them together to get the next one, and that is why it is recursive.

32.1 Convergence and Divergence

One thing that we like to know about sequences is if they converge or diverge. We talked about convergence and divergence before with improper integrals, but let's refresh our memory. If something **CONVERGES**, then if we take the limit to infinity, we will go to one specific number. If something doesn't converge, it **DIVERGES**. There is two kinds of divergence. If the limit as we go to infinity doesn't exist, we just diverge. If the limit as we go to infinity goes to positive or negative infinity, then we diverge to infinity.

Here is a theorem we have for sequences:

If $\lim_{x\to\infty} f(x)$ exists, then the sequence $a_n = f(n)$ converges to the same limit:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

Here are some examples with actual sequences.

1. $a_n = \dfrac{n+4}{n+1}$ converges to 1 because

$$\lim_{n\to\infty} \frac{n+4}{n+1} = \lim_{x\to\infty} \frac{x+4}{x+1} = 1$$

2. $a_n = \cos n$ diverges because as we go to infinity the cosine function does not settle on one value.

$$\lim_{n\to\infty} \cos n = \lim_{x\to\infty} \cos x = \text{Does Not Exist}$$

3. $a_n = 3(2^n)$ diverges to infinity because

$$\lim_{n\to\infty} 3(2^n) = \lim_{x\to\infty} 3(2^x) = \infty$$

The last example above is the first special type of sequence we are going to look at. It is called a **GEOMETRIC SEQUENCE**. A geometric sequence is a sequence in the following form:

$$a_n = cr^n$$

where c is a constant and r is a rate. The special thing about this kind of sequence is this:

$$\lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if } 0 \le r \le 1 \\ c & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

Here is another theorem for sequences. It is called squeeze theorem.

Let c_n, a_n and b_n be sequences such that for some number M, $c_n \le a_n \le b_n$ for $n > M$ and $\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = L$. Then $\lim_{n\to\infty} a_n = L$.

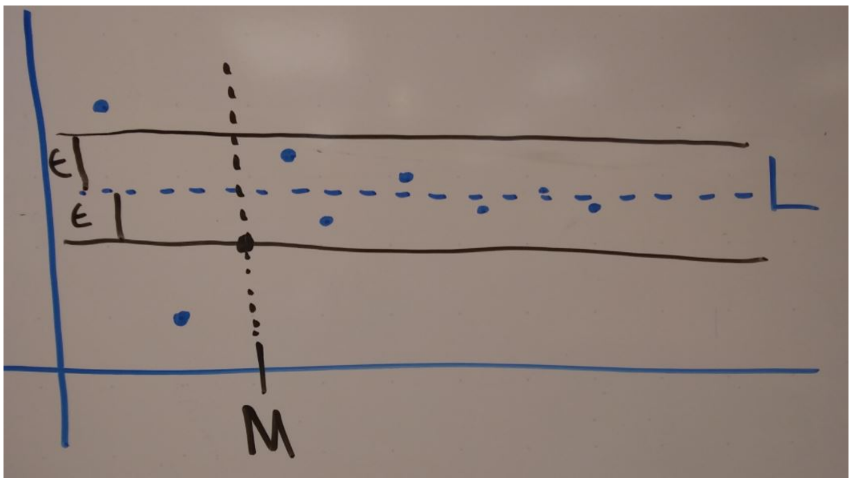
Looking at that theorem, you might be wondering what that M is. I am going to give you a more mathematical definition of convergence, and then it might make more sense.

We say a sequence (a_n) converges to a limit L, and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L$$

if, for every $\varepsilon > 0$, there is a number M such that $|a_n - L| < \varepsilon$ for all $n > M$.

This is basically saying that for any number that we give the name ε (it could be huge, but generally we want it to be small), there is a number M where any n that we plug into our sequence a_n, the term we get out will be closer to L than the distance distance between ε and L. In other words, $|a_n - L| < |L - \varepsilon|$ when $n > M$.



As you can see in the picture the dashed horizontal line is the limit of the sequence. ε is just a number with which we get the horizontal lines $y = L \pm \varepsilon$. Since our sequence converges to L, there is a point M where every a_n when $n > M$ falls between $y = \pm\varepsilon$.

Now let's try the algebra part of the definition. If we use the sequence from before, $a_n = \dfrac{n+4}{n+1}$, then by the definition of convergence we can say this:

$$\begin{aligned}
\left|\frac{n+4}{n+1} - 1\right| &< \varepsilon \\
\frac{3}{n+1} &< \varepsilon \\
3 &< \varepsilon(n+1) \\
\frac{3}{\varepsilon} - 1 &< n
\end{aligned}$$

From this inequality, given any ε, we can find an M $|a_n - L| < |L - \varepsilon|$ for every $n > M$. So if $\varepsilon = .005$, $M = \dfrac{3}{\varepsilon} - 1 = 599$. If we find $a_{600} = 1.004992$ which is closer to 1 than 1.005.

This is most everything about sequences, but there are a few more theorems in your book. Make sure you take the time to go over them a bit (most of them are common sense).

33 Series

A series is a sequence that we add all of the terms together. Basically we are going to add an infinite amount of numbers together. The question we are interested in is if all of these terms add up to a number or do they go to infinity. Do they converge or diverge? This might be a weird thing to think about, and you might be asking, "How can an infinite amount of numbers add to get a finite number?"

Let's look at the sequence $a_n = \left(\frac{1}{2}\right)^n$. Looking at the first few terms and adding them up, we will get $\frac{1}{2} + \frac{1}{4} + \frac{1}{16} = 0.8125$. If we think about it, each consecutive term will get us only half way to 1, and if we are only getting half way to one each time, we will never reach 1. This means our series will converge to 1. Therefore,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$$

Just because a sequence converges, does not mean its series will, if we look at $a_n = \frac{1}{n}$, the sequence will converge to 0, but if we add the terms together, we will get $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$. You might be able to see that this sum will keep growing even though it will grow very slowly. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \text{ Divergent.}$$

This is great, but how can we know if a series converges when they are more complex? That is exactly what you are going to learn over the next few weeks.

34 Geometric Series

A geometric series is the first kind of series we are going to look at.

A geometric series is an infinite sum of the form

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n.$$

The value of r is called the ratio.

In a geometric series, if $|r| < 1$ and n starts at 0, then the series converges to $\dfrac{a}{1-r}$. Thus, if $|r| < 1$,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

However, if n starts at some number other than 0 (call it M), then if $|r| < 1$,

$$\sum_{n=M}^{\infty} ar^n = \frac{ar^M}{1-r}.$$

If $|r| > 1$, then a geometric series diverges.

35 Telescoping Series

A telescoping series is a series where a lot of the middle terms cancel out. If we look at $a_n = \dfrac{1}{n(n+1)}$, we can use partial fractions and get $a_n = \dfrac{1}{n} - \dfrac{1}{n+1}$ thus,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

If we plug in the first few terms, we get

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$$

If we continue this, we will get

$$\frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N}} + \cancel{\frac{1}{N}} - \frac{1}{N+1} = 1 - \frac{1}{N+1}$$

Since we know this, we can find this limit:

$$\lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$$

There is no set rule for telescoping series, but if you recognize a series is telescoping then you can do this process with it.

36 Integral Test

The integral test for series basically turns a series into an improper integral. There are a few requirements that you have to check before you can use the integral test. Assume our series is

$$\sum_{n=1}^{\infty} a_n.$$

Requirements for integral test:

1. Change your sequence a_n into a function $f(n)$ (this doesn't actually change anything except the domain is $\mathbb{R}$ instead of $\mathbb{N}$).
2. $f(n)$ must be continuous
3. $f(n)$ must be positive
4. $f(n)$ must be decreasing ($f'(n) < 0$)
5. $f(n)$ must be integrable

If all of this criteria is met, then we can proceed with the integral test which is:

1. If $\int_1^\infty f(n)\, dn$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\int_1^\infty f(n)dn$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Let's look at an example.

We have already seen this series before, and I told you that it was divergent. I explained it intuitively, but in math that isn't always good enough.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Assume $f(n) = \frac{1}{n}$, and let's check the criteria from above

1. $f(n)$ is continuous

2. $f(n)$ is positive

3. $f(n)$ is decreasing ($f'(n) = \frac{-1}{n^2} < 0$)

4. $f(n)$ is integrable

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n} = \int_1^\infty \frac{1}{n}\, dn = \ln n \Big|_1^\infty = \text{divergent}$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This series actually has a special name, "the harmonic series."

37 Comparison Test

The comparison test is not too complicated. That however, does not mean it is easy to recognize or easy to use all the time. I will say this one definitely takes some practice to use. I am going to state the exact theorem for the comparison test because there are definitely some details.

Assume there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for every $n \geq M$.

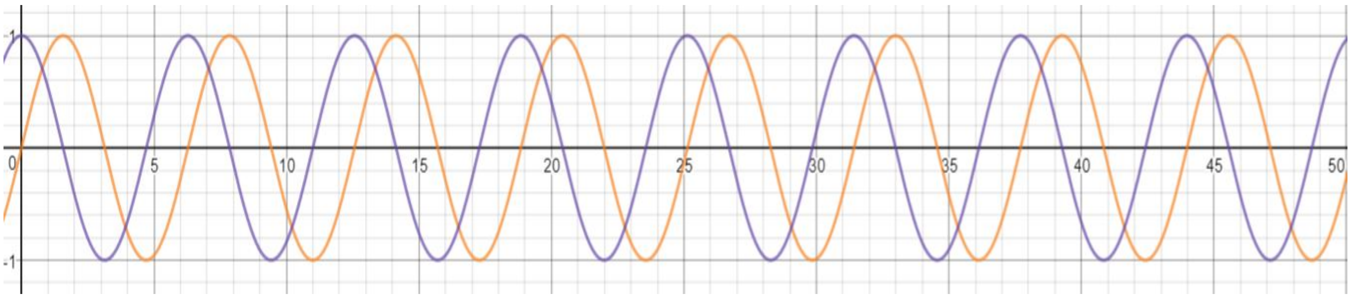
1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

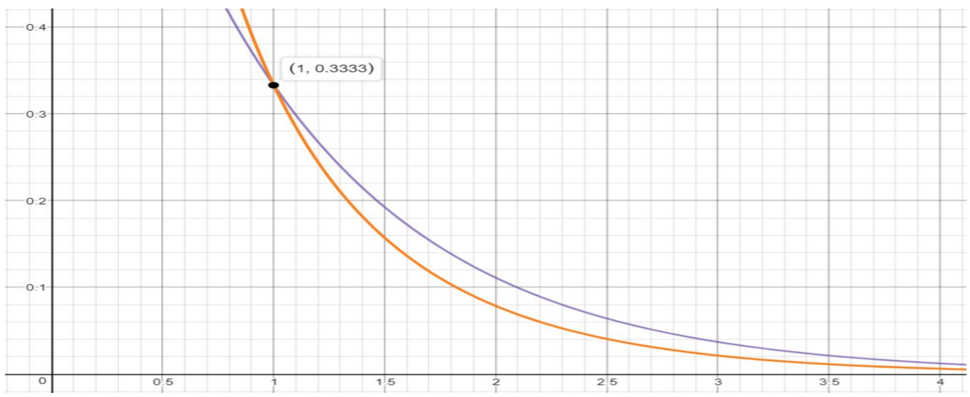
This theorem is saying that if you have a smaller series and larger series, if the larger converges, then the smaller one converges.

Similarly, if you have a smaller series and larger series, if the smaller diverges, then the larger one diverges.

What is all this stuff about M though? The M is saying that since the partial sum of the series is finite, as long as one sequence is larger than the other past a certain, and they don't keep switching, you are good to compare. Here are 2 pictures.



Above is $\sin x$ and $\cos x$. As you can see in this picture, there is no point M where every $x \geq M$ one function is always bigger than the other. They will keep alternating positions.



These two functions are $f(x) = \dfrac{1}{\sqrt{x}3^x}$ (orange), and $g(x) = \dfrac{1}{3^x}$ (purple). As you can see, if we let $M = 1$, if we can see that for every $x \geq 1$, $g(x) \geq f(x)$. Thus we can compare the 2 series $\sum_{n=1}^{\infty} f_n$ and $\sum_{n=1}^{\infty} g_n$.

Since $\sum_{n=1}^{\infty} \dfrac{1}{3^n}$ is a geometric series with an $r < |1|$, it converges. Since $\sum_{n=1}^{\infty} f_n \leq \sum_{n=1}^{\infty} g_n$. $\sum_{n=1}^{\infty} f_n$ converges.

Let's pretend $\sum_{n=1}^{\infty} g_n$ diverged. This would mean we have no information about $\sum_{n=1}^{\infty} f_n$ because just because a bigger sequence diverges, doesn't mean the smaller sequence has to diverge. Basically, if you want to prove something converges, make a series that is bigger and show it converges. If you want to prove something diverges, make a series that is smaller and show it diverges.

38 Alternating Series Test

An alternating series is named for how it alternates between positive and negative terms. An alternating series will look something like these:

$$\sum_{n=1}^{\infty} (-1)^n (b_n) \text{ or } \sum_{n=1}^{\infty} (-1)^{n\pm 1} (b_n).$$

Basically the difference between these series is that in the first one, the terms will be

$-b_1+b_2-b_3+\cdots+(-1)^n b_n$, and the second will be $b_1-b_2+b_3\cdots+(-1)^{n+1}b_n$. The exponent on the alternating part only changes which terms are positive and which are negative.

To deal with these kinds of series we have the alternating series test. Just like our other tests, this test has some criteria we need to check. Assume our series is $\sum\limits_{n=1}^{\infty}(-1)^n b_n$ or $\sum\limits_{n=1}^{\infty}(-1)^{n\pm1}b_n$.

1. b_n must be decreasing ($b_{n+1}<b_n$ for all $n\geq 1$)
2. $\lim\limits_{n\to\infty} b_n = 0$

If we meet all of this criteria, $\sum\limits_{n=1}^{\infty}(-1)^n b_n$ and $\sum\limits_{n=1}^{\infty}(-1)^{n\pm1}b_n$ converge. Let's look at the alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

1. $\frac{1}{n}$ is decreasing because if we let $f(x)=\frac{1}{x}$, then $f'=-\frac{1}{x^2}$ which will always be negative.
2. $\lim\limits_{n\to\infty}\frac{1}{n}=0$.

Therefore, the alternating harmonic series converges.

39 Absolute and Conditional Convergence

When we talk about alternating series, sometimes we like to know if our sequence converges absolutely or just conditionally. If an alternating series $\sum\limits_{n=1}^{\infty}(-1)^n b_n$ converges, then we can check to see if $\sum\limits_{n=1}^{\infty}|(-1)^n b_n|$ (this series is equivalent to $\sum\limits_{n=1}^{\infty} b_n$) converges. If $\sum\limits_{n=1}^{\infty}|(-1)^n b_n|$ converges, then we can say the series is absolutely convergent, if not, our series is conditionally convergent.

Let's look at the following two series.

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$$

By the alternating series test, both of these converge (check if you want to practice).

Now let's see if they converge absolutely.

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n}\right| \qquad \text{and} \qquad \sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n^2}\right|$$

We know we can rewrite these as

$$\sum_{n=1}^{\infty}\frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{1}{n^2}$$

These series are easy because the first is the harmonic series (divergent) and the second is a geometric series where $r=\frac{1}{2}$ (convergent). This means $\sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{n}$ is conditionally convergent and $\sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$ is absolutely convergent.

40 Ratio Test

The ratio test is very helpful on series that contain factorials, exponential factors like $n!$ or 3^n. Assume our series is $\sum_{n=1}^{\infty} a_n$. All we are going to do is first check that $a_n > 0$ for all $n \geq 1$, and second find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

1. If $L < 1$, the series converges.
2. If $L > 1$, the series diverges.
3. If $L = 1$ the test is inconclusive (do a different test).

Let's see if $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or diverges (let $a_n = \frac{2^n}{n!}$).

By plugging in a few numbers, we can see that a_n is decreasing.

Now let's find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$$

Using algebra we can rewrite this as

$$\lim_{n\to\infty} \frac{2}{n+1} = 0$$

Therefore, $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

41 Root Test

The root test works best when you have exponential sequence like $(a_n)^n$. Algebraically, $\lim_{n\to\infty} (a_n)^n = \lim_{n\to\infty} \sqrt[n]{(a_n)^n} = \lim_{n\to\infty} a_n = L$.

1. If $L < 1$, the series converges.
2. If $L > 1$, the series diverges.
3. If $= 1$, the test is inconclusive.

Assume our series is $\sum_{n=1}^{\infty} \left(\frac{2n}{3n-1}\right)^n$

$$\lim_{n\to\infty} \left(\frac{2n}{3n-1}\right)^n = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{3n-1}\right)^n} = \lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3}$$

Since $L = \frac{2}{3} < 1$, $\sum_{n=1}^{\infty} \left(\frac{2n}{3n-1}\right)^n$ converges.

42 Summary of Convergence Tests

1. **Divergence Test:** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

2. **Geometric Series:** Let $c \neq 0$. If $|r| < 1$, then

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \cdots = \frac{c}{1-r}$$

$$\sum_{n=M}^{\infty} cr^M = c + cr + cr^{M+1} + cr^{M+2} + \cdots = \frac{cr^M}{1-r}$$

   If $|r| \geq 1$, then the geometric series diverges.

3. **Telescoping Series:** Case by case. Let $\sum a_n$ be a telescoping series. Assume $a_1, a_2, \ldots a_n$ are the first terms that don't cancel, and b_n is the form of the final terms that don't cancel, then $\sum a_n = a_1 + a_2 + \cdots + a_n + \lim_{n\to\infty} b_n$.

4. **Integral Test:** Let $a_n = f(n)$ where f is a positive, decreasing, and continuous function of x for $x \geq 1$.

   i) If $\int_1^\infty f(x)$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   ii) If $\int_1^\infty f(x)$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

5. Assume there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for every $n \geq M$.

   i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

6. **Limit Comparison Test:** Let $\{a_n\}$ and $\{b_n\}$ be **positive** sequences. Assume that the following limit exists

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

   - If $L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
   - If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.
   - If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

7. **Power Series (p-Series):** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

8. **Alternating Series Test:** Assume that $\{b_n\}$ is a positive sequence that is decreasing and converges to 0:

$$b_1 > b_2 > b_3 > \cdots > 0 \text{ and } \lim_{n\to\infty} b_n = 0$$

Then the following alternating series converges:

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - \dots$$

Furthermore,

$$0 < S < b_1.$$

9. **Ratio Test:** Assume the following limit exists:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

i) If $\rho < 1$, then $\sum a_n$ converges absolutely.
ii) If $\rho > 1$, then $\sum a_n$ diverges.
iii) If $\rho = 1$, then the test is inconclusive.

10. **Root Test:** Assume the following limit exists:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

i) If $L < 1$, then $\sum a_n$ converges absolutely.
ii) If $L > 1$, then $\sum a_n$ diverges.
iii) If $L = 1$, then the test is inconclusive

43 Power Series

The next kind of infinite series we are going to look at is called a power series.

A power series with center c (c is a constant) is written as

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Like any other series we have been looking at, we want to know when it converges or diverges. Notice that a power series is a **FUNCTION** of x this means for certain values of x, we could have a convergent series, and for others we could have a divergent series.

If we think of the time when $x = 0$, we get this:

$$f(0) = \sum_{n=0}^{\infty} a_n (-c)^n$$

This looks really similar to a geometric series (if you let $a_n = c$ and $-c = r$) which is $\sum_{n=0}^{\infty} cr^n$. We know a geometric series is convergent if $|r| < 1$ so we need to find all x values that make $|x-c| < 1$. We call this the **radius of convergence**. We also know we can split $|x-c| < 1$ up into $1 < x-c < 1$, and we would call this the **interval of convergence**. Let's look at an example.

$$\sum_{n=0}^{\infty} \frac{x^n}{n \cdot 5^n}$$

Looking at this power series, it looks like we should use the **ratio test** $\left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \right)$ to see if it converges or diverges (this is what you should do for **EVERY** power series). We still have an x though...what should we do?

I am glad you noticed! At this point, we are going to leave it as x. Let's see what happens.

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)\cdot 5^{n+1}}}{\frac{x^n}{n \cdot 5^n}} \right|$$

Let's multiply by the reciprocal and see what cancels.

$$\lim_{n\to\infty} \left| \frac{x^{\cancel{n+1}}}{(n+1)\cdot 5^{\cancel{n+1}}} \cdot \frac{n \cdot \cancel{5^n}}{\cancel{x^n}} \right|$$

Let's simplify now.

$$\lim_{n\to\infty} \left| \frac{x \cdot n}{(n+1)\cdot 5} \right|$$

To find this limit, distribute the 5 in the denominator and let's multiply our fraction by $\left(\frac{\frac{1}{n}}{\frac{1}{n}} \right)$

$$\lim_{n\to\infty} \left| \frac{x \cdot n}{5n+5} \cdot \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{5+\frac{1}{n}} \right|$$
$$= \left| \frac{x}{5} \right| = \rho$$

We know when we use the ratio test

- If $\rho < 1$, then $\sum a_n$ converges absolutely.
- If $\rho > 1$, then $\sum a_n$ diverges.
- If $\rho = 1$, then the test is inconclusive.

So let's solve for x now.

$$-1 < \frac{x}{5} < 1$$
$$-5 < x < 5$$

This gives us a radius of convergence of $|x| < R = 5$, but this is not necessarily the interval of convergence.

As I just said, if $\rho = 1$, then the test is inconclusive. So that means we have to test $x = \pm 5$ individually.

$x = 5$

$$\sum_{n=0}^{\infty} \frac{5^n}{n \cdot 5^n} = \sum_{n=0}^{\infty} \frac{1}{n} \Rightarrow \text{harmonic series: divergent}$$

$x = -5$

$$\sum_{n=0}^{\infty} \frac{(-5)^n}{n \cdot 5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \Rightarrow \text{alternating harmonic series: convergent}$$

This means out interval of convergence is $-5 \le x < 5$

The radius of convergence gives us the segment $(c-R, c+R)$, and if x is in this segment then the series absolutely converges. The interval of convergence can include the endpoints.

This is how you will find the interval of convergence for power series.

1. Use ratio test to find radius of convergence.
2. Test the endpoints to find the interval of convergence.

44 Differentiating Power Series

Remember power series are functions ($f(x) = \sum\limits_{n=1}^{\infty} a_n(x-c)^n$). So what is stopping us from finding their derivative? Let $f(x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n \cdot 5^n}$, and $b_n = \frac{x^n}{n \cdot 5^n}$. How can we find $f'(x)$? If we think about it, let's see if we find the first few terms of our series if that will help.

$$f(x) = \frac{x}{5} + \frac{x^2}{50} + \frac{x^3}{375} + \cdots$$

To find $f'(x)$, we would have to find the derivative of each term.

$$f'(x) = \frac{1}{5} + \frac{x}{25} + \frac{x^2}{125} + \cdots$$

If we find $\frac{d}{dx}b_n$, would that be the same thing? Let's try.

$$\frac{d}{dx}\left(\frac{x^n}{n \cdot 5^n}\right)$$

Remember we are differentiating with respect to x which means we treat n as a constant.

$$\frac{d}{dx}b_n = \frac{d}{dx}\left(\frac{x^n}{n \cdot 5^n}\right) = \frac{nx^{n-1}}{n \cdot 5^n} = \frac{x^{n-1}}{5^n}$$

Now let's plug that back into our series.

$$\sum_{n=1}^{\infty} \frac{d}{dx}b_n = \sum_{n=1}^{\infty} \frac{x^{n-1}}{5^n}$$

If we test our first few terms we will get the following:

$$\frac{1}{5} + \frac{x}{25} + \frac{x^2}{125} + \cdots$$

. We know that's $f'(x)$ when we went term by term so we can say

$$f'(x) = \sum_{n=1}^{\infty} \frac{d}{dx}b_n = \sum_{n=1}^{\infty} \frac{x^{n-1}}{5^n}$$

In other words is isn't hard to differentiate a power series.

$$f'(x) = \sum_{n=1}^{\infty} \frac{d}{dx}b_n$$

45 Integrating Power Series

Whatever we differentiate, we can always integrate. If we let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 5^n}$, and $b_n = \frac{x^n}{n \cdot 5^n}$. Let's find $\int f(x)$.

$$f(x) = \frac{x}{5} + \frac{x^2}{50} + \frac{x^3}{375} + \cdots$$

To find $\int f(x)$, we would have to find the integral of each term.

$$\int f(x) = \frac{x^2}{10} + \frac{x^3}{150} + \frac{x^4}{1500} + \cdots$$

Let's find $\int b_n \, dx$ and see if that works.

$$\int b_n \, dx = \int \frac{x^n}{n \cdot 5^n} \ = \frac{1}{n+1} \cdot \frac{x^{n+1}}{n \cdot 5^n} + C = \frac{x^{n+1}}{(n+1) \cdot n \cdot 5^n} + C$$

Now plug in $\int b_n\, dx$ into the series and put C out side our series.

$$\sum_{n=1}^{\infty} \int b_n\, dx = \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1) \cdot n \cdot 5^n}$$

This gives us these first few terms:

$$C + \frac{x^2}{10} + \frac{x^3}{150} + \frac{x^4}{1500} + \cdots$$

We know that's $\int f'(x)$ when we went term by term so we can say

$$\int f(x) = \sum_{n=1}^{\infty} \int b_n = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1) \cdot n \cdot 5^n}$$

In other words is isn't hard to integrate a power series.

$$\int f(x) = C + \sum_{n=1}^{\infty} \int b_n$$

46 Power Series Representing a Function

Given a finite number of terms power series approximate a function. However if we find a infinite number of terms, our power series becomes a function. Let's look at $tan^{-1}x$. We know $\frac{d}{dx}\tan^{-1} = \frac{1}{1+x^2}$. This means if we can represent $f(x) = \frac{1}{1+x^2}$ with a power series, we can integrate and find a representation for $\tan^{-1} x$. Let's look at the process to find a power series that can represent $f(x) = \frac{1}{1+x^2}$.

We know a geometric series looks like $\sum\limits_{n=0}^{\infty} ar^n$, and if $|r| < 1$, a geometric series converges to $\frac{a}{1-r}$. If we let $a = 1$ and $r = -x^2$, we get this:

$$f(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} \int (-1)^n (x^{2n})$$

Based on our $a = 1$ and $r = -x^2$, if $|x^2| < 1$ (which is the same as $|x| < 1$), our series will converge to $\dfrac{1}{1+x^2}$ which is the derivative of $tan^{-1}x$. This means if we integrate our power series, and solve for our constant, we will get $\tan^{-1} x$. Let's try that.

$$\int f(x)\, dx = \sum_{n=0}^{\infty} \int (-1)^n (x^{2n})\, dx$$

We can say this is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

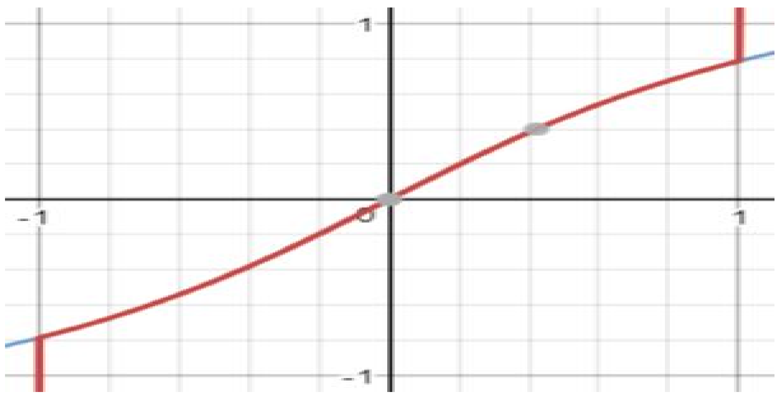
This will give us these first few terms:

$$C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots$$

As you can see I have placed my constant "C" at the beginning just because placing it at the "end" of a infinite list of numbers just doesn't work. now if we let $x = 0$, we will be left with $tan^{-1}(0) = C$ because every term in our series has an x in the numerator, and thus all of them will be 0. Therefore, since $tan^{-1}(0) = 0$, we know $C = 0$. This means we can say

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Remember this series only converges when $|x| < 1$. The following is a picture of the series (red) versus the function (blue)



As you can see, we can represent functions with power series, but we are limited by the interval of convergence (in this case, $-1 < x < 1$).

Another important power series that represents a function is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The coolest part of this power series is that the radius of convergence is $R = \infty$ which means we can plug in any x! If you want to, you can integrate or differentiate this power series, and you will see that once you simplify, you will be back to the original series.

47 Special Power Series

There are 2 specific types of power series that are generally used to represent functions. They are Taylor Series and Maclaurin Series.

A Taylor Series is a power series centered at a that follows this formula:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Maclaurin Series is a Taylor Series that is centered at 0. A Maclaurin Series follows this formula:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

Note that $f^{(n)}(x)$ is the n^{th} derivative evaluated at x.
Let's look at the Maclaurin Series in detail first.

47.1 Maclaurin Series

Maclaurin Series are used a little more than Taylor Series. Let's look at how to make a Maclaurin Series for $\sin x$.

Looking at the numerator, we need to find the nth derivative evaluated at 0.

$$\begin{aligned} f(0) &= \sin(0) = 0 \\ f'(0) &= \cos(0) = 1 \\ f''(0) &= -\sin(0) = 0 \\ f'''(0) &= -\cos(0) = -1 \end{aligned}$$

As we know, this pattern will repeat forever. The pattern we have going so far is 0, 1, 0, -1. If we think about this pattern, it will mean that even term will have a numerator equal to 0, and therefore the term will equal 0. This means we only have to focus on the odd terms. The non-zero terms alternate, and the first non-zero term is positive, so when we plug in $n = 0$, we need to have a positive term come out. Let's have the numerator be $(-1)^n$.

As we know, the even terms will all be 0 so we don't want our series to give a number on even terms so let's make our denominator be $(2n+1)!$. We know this will only turn out odd number terms. Similarly, if we raise x to the $2n+1$ power, we will get only the odd terms (x^1, x^3, $x^5, \dots$). Putting all of this information together, we will find the following Maclaurin Series:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

The first few terms are as follows:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

As you can see this fits the information we had. All even terms are 0, and each odd term alternates positive and negative starting as positive.

We can also find a Maclaurin Series for $\cos(x)$ using the same process. Let's look at the derivatives evaluated at 0.

$$\begin{aligned} f(0) &= \cos(0) = 1 \\ f'(0) &= -\sin(0) = 0 \\ f''(0) &= -\cos(0) = -1 \\ f'''(0) &= \sin(0) = 0 \end{aligned}$$

Now as we can see, odd terms will be 0 and the even terms will alternate from positive to negative. This means our series will look like this:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

The first few terms are as follows:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Both of these series have a radius of convergence of $R = \infty$.

I also mentioned earlier that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and we get that again using the Maclaurin Series formula.

47.2 Taylor Series

Looking at how a Maclaurin Series works, why would we ever need to center our series somewhere other than 1? If you think for a long enough time, $\ln x$ is a function we need to center somewhere other than 0. If you didn't think of a function that we would have to center somewhere other than 0, what are some of the problems with centering $\ln x$ at 0?

- $\ln x$ does not exist at 0
- None of the derivatives of $\ln x$ exist at 0 (even if one didn't we would have a problem)

For these reasons, we would have to center our series somewhere the function and derivative exists. Let's center $\ln x$ at $a = 1$. The formula for a Taylor series is this:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We still need our derivatives evaluated at 1 now.

$$\begin{aligned} f(1) &= ln(1) \\ f'(1) &= x^{-1} = 1 \\ f''(1) &= -x^{-2} = -1 \\ f'''(1) &= 1 \cdot 2x^{-3} = 2 \\ f^{(4)}(1) &= 1 \cdot 2 \cdot 3x^{-4} = -6 \end{aligned}$$

I don't know about you, but I see a pattern forming here. I am seeing that $f^{(n)} = (-1)^{n-1}(n-1)!(x^{-n})$ and since $x = 1$, we can say $f^{(n)} = (-1)^{n-1}(n-1)!$. We can see that $\ln(1)$ doesn't follow our formula so we will say $\ln(1) + \sum_{n=1}^{\infty} a_n$, but we also know $\ln(1) = 0$ so we can actually leave it off. Let's plug that into our Taylor series formula.

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n$$

We can cancel our factorials and simplify to get this:

$$\ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$$